POLYNOMIAL TAPER EQUATIONS THAT ARE COMPATIBLE WITH TREE VOLUME EQUATIONS

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(Received for publication 19 March 1975)

ABSTRACT

The theory is discussed of individual tree compatible taper equations which predict the diameter along the stem as a function of tree height, diameter at breast height, and length, and which can be integrated to give a total volume equation equal to an existing volume equation. Existing non-linear forms of compatible taper equations were estimated for **Pinus radiata** D. Don but were unable to illustrate the neiloid shape of the butt, common in old crop **P. radiata**. Polynomial compatible taper equations were developed to provide greater flexibility in defining tree shape. These had standard errors of estimate of diameter of 1.4 cm for young crop and 2.4 cm for old crop radiata pine, and they had all the desirable characteristics of compatible taper equations.

INTRODUCTION

In a series of papers Demaerschalk (1971, 1972a, 1972b, 1973a, 1973b) and Munro and Demaerschalk (1974) introduced taper equations for the stems of trees which, when integrated to give a sectional volume equation, calculated a total volume equal to that of an existing total volume equation. The equations were termed "Compatible Taper Equations". In their basic form they predict the square of diameter along the stem as a function of length from the tip, diameter at breast height and total height of the tree. Their usefulness is apparent where total volume equations already exist and will continue to be used in the future. Because of their properties, compatible taper equations are ideal for use in log cutting-pattern optimization procedures, such as that by Pnevmaticos and Mann (1971), or in calculating the proportions of volume in the various utilization categories, as required by forest inventory. It was for these two reasons that the present study investigated compatible taper equations for radiata pine (*Pinus radiata* D. Don) in Kaingaroa Forest, and extended their theory.

DEVELOPMENT AND APPLICATION OF NEW THEORY

Existing Theory

The equations suggested by Demaerschalk were non-linear and had several important and desirable properties.

1. They can be integrated to calculate sectional volumes, and the total volume calculated is compatible with an existing total volume equation.
2. The calculated diameter is strictly non-negative over the interval from stump to total height.
3. The function of d^2 is a monotonic non-decreasing function. This implies that the calculated volume of a log cut from the top of the tree will never be greater than one of equal length cut below it.
4. The diameter at the tip is 0.
5. The simplest of the equations can readily be arranged to give merchantable length as a function of merchantable diameter, merchantable volume as a function of length or diameter.

The standard errors (S.E.) of estimate of diameter* along the stem are small—as good as, if not better than, other non-compatible taper equations which often do not have all of the above properties. To recapitulate the existing theory, and in extending the theory the following notation will be used.

Let V = total volume inside bark (m^3)

D = diameter breast height (dbh) outside bark (cm)

H = total height (m)

l = distance *from the tip* of the tree (m)

d = diameter inside bark at l m from the tip (cm)

V_m = volume inside bark *from the tip* of the tree to a point l with diameter inside bark d

$K = (\pi/4)\ 10^{-4}$

The general expression of a non-linear taper equation is

$$d^2 = \frac{(p+1)}{K}\ \frac{V}{H} \left[\frac{l}{H} \right]^p \qquad (1)$$

where p is a "free" parameter to be estimated from taper data.

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* The S.E. of estimate of diameter was calculated by Demaerschalk, and throughout this paper, from the formula

$$SE\ (d_e) = [\Sigma\ (d - \hat{d})^2/(n - m - 1)]^{\frac{1}{2}}$$

where d, $\hat{d}$ are the actual and predicted diameters inside bark,

n = no. of observations, and
m = no. of "free" parameters in the taper equation.

It should be noted that V can be calculated from any existing volume equation. That this is compatible should be clear, for multiplying by K and integrating between the limits of 0 and h gives

$$V_m = \int_0^h \left(\frac{\pi}{4}\right) 10^{-4}\ d^2\ dl$$

$$= \int_0^h K \frac{(p+1)}{K} \frac{V}{H} \left[\frac{l}{H}\right]^p dl$$

$$= \left[V \left[\frac{l}{H}\right]^{p+1} \right]_0^h \tag{2}$$

When total volume is required, h = H,

$$V_m = V$$

If the function to estimate V is formed from the sum of independent functions e.g. $V = f_1(D,H) + f_2(D,H)$ then additional "free" parameters can be introduced to the taper equation and estimated from taper data, i.e.:

$$d^2 = \frac{(p+1)}{K} \frac{f_1(D,H)}{H} \left[\frac{l}{H}\right]^p + \frac{(q+1)}{K} \frac{f_2(D,H)}{H} \left[\frac{l}{H}\right]^q \tag{3}$$

The principles should be quite clear.

The Fit of the Non-Linear Taper Function to Kaingaroa Data

Radiata pine in Kaingaroa Forest is divided into two crop types, old crop planted prior to 1940 and young crop. The old crop especially contains many malformed stems but this study is confined to the taper of "normal" stems, that is, single-leader straight trees with no large deformities. Large numbers of trees have been sectionally measured over the past 20 years and the measurements of some 914 old crop and 353 young crop trees were selected for this study. Total volume equations were estimated for both old and young crop, and were found to be significantly different. Duff's (1954) volume tables were not used as their data were entirely old crop and had few trees greater than 45 m in height and 75 cm dbh. The selection and distribution of data for this study, and the volume equation analysis will be described elsewhere (Goulding and Murray, in prep.). Briefly, the old crop data consisted of trees with a range of dbh of 15-102 cm, and in height of 14-56 m; young crop with a range in dbh of 15-64 cm, and height of 12-41 m.

The total volume equation selected for young crop radiata was

$$V = 0.25934\ D^2H\ 10^{-4} + 0.13407\ H\ 10^{-2} \tag{4}$$

$$SE(V) = 0.2423\ D^2H\ 10^{-5}\ \text{cu m.}$$

The parameters p and q of equation (3) were calculated, where $f_1(D,H) = 0.25934\ D^2H\ 10^{-4}$ and $f_2(D,H) = 0.13407\ H\ 10^{-2}$. However, one of the parameters, q, was not significantly different from 0 at the 95% level, as was the case for a similar equation in the old crop. Therefore the parameter p in equation (1) was estimated alone.

$$d^2 = \frac{2.5466}{K} \frac{V}{H} \left[\frac{l}{H} \right]^{1.5466}$$

$$V_m = V \left[\frac{l}{H} \right]^{2.5466}$$

$$S.E.(p) = 0.005022 \quad S.E.(d^2) = 66.56$$

The S.E. of the estimate of d, $SE(d_e)$, was calculated directly to be 1.40 cm. Fig. 1 illustrates the taper curves for various trees.

Although the $SE(d_e)$ is very low, Fig. 1 shows the major limitation of the non-linear taper equation. The shape makes no allowance for butt-swell which is quite noticeable in radiata. Moreover, even when 2 "free" parameters were calculated the shape was the same, indicating that the class of functions was not flexible enough to account for any variations from the basic shape.

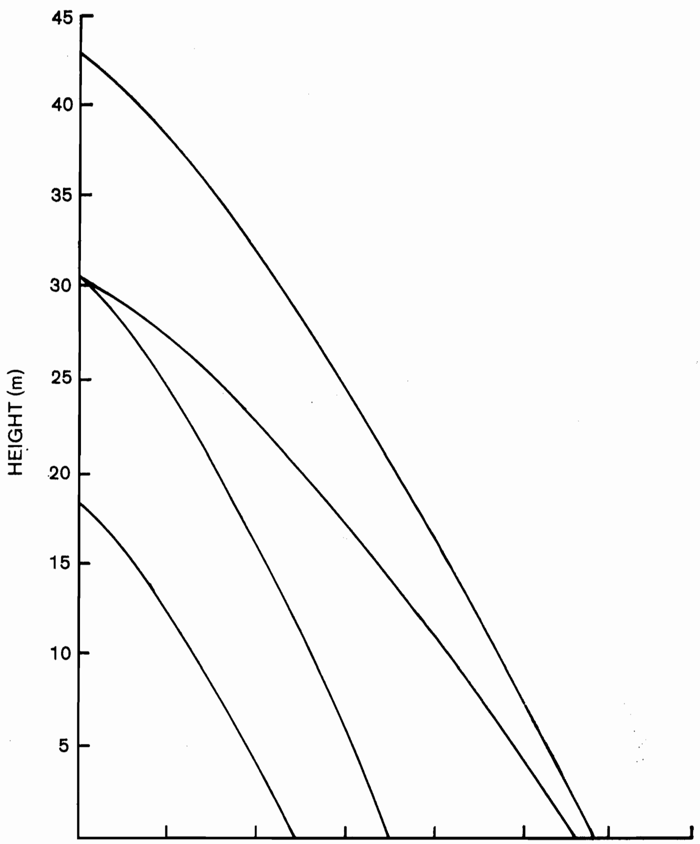

FIG. 1—Young crop non-linear taper curves for selected values of dbh and height.

New Theory

There exist many other forms of compatible taper equations. For example, it is possible to modify the taper equation suggested by Kozak *et al.* (1969a, 1969b) to make it compatible with the volume equation $V = a_o + a_1 D^2 H$ where a_o, a_1 are the estimated regression coefficients. Taking the original taper equation —

$$\frac{d^2}{D^2} = b_o + b_1 \left[\frac{l}{H}\right] + b_2 \left[\frac{l}{H}\right]^2 \tag{6}$$

$$\begin{aligned} V &= \int_o^H K\, d^2\, dl \\ &= K\, D^2 \left[b_o H + (b_1 H)/2 + (b_2 H)/3\right] \\ &= a_1{}'\, D^2 H \end{aligned}$$

This lacks only the constant term a_o which can be supplied by the addition to the taper equation of the term b_3/D^2H. The modified taper equation is thus —

$$\frac{d^2}{D^2} = b_o + b_1 \left[\frac{l}{H}\right] + b_2 \left[\frac{l}{H}\right]^2 + \frac{b_3}{D^2 H} \tag{7}$$

where $b_3 = a_o/K$
$b_o = a_1/K - b_1 - b_2$

This equation can be solved by a conditioned linear least squares routine. However, when the equation was fitted to the data, despite a low $SE(d_e)$, the basic shape was deficient in the region near the tip of the tree, failing to have 0 diameter at the tip and often having negative values of d^2 for various combinations of D and H.

In general, a taper equation is required of the form

$$d^2 = \frac{V}{KH} f \left[\frac{l}{H}\right] \tag{8}$$

where $f\left[\frac{l}{H}\right]$ is a polynomial in $\left[\frac{l}{H}\right]$ such that the coefficients of the polynomial of degree n (b_i) satisfy

$$\sum_{i=o}^{n} \frac{b_i}{i+1} = 1$$

An alternative form is

$$d^2 = \frac{V}{K} \left[f\left[\frac{l}{H}\right] + \frac{2l}{H^2} \right]$$

where $\sum_{i=o}^{n} \frac{b_i}{i+1} = 0$

By taking the polynomial $f\left[\frac{l}{H}\right]$ to have zero intercept, the taper equation will have zero diameter at the tip of the tree. Both equations (8) and (9) are compatible taper equations, and can be fitted to any order of polynomial required by the tree shape.

To illustrate with equation (8)

$$d^2 = \frac{V}{KH}\left[b_1\left(\frac{l}{H}\right) + b_2\left(\frac{l}{H}\right)^2 + \ldots + b_n\left(\frac{l}{H}\right)^n\right] \qquad (10)$$

$$Vm = K\int_0^h d^2 \,.\, dl$$

$$= \frac{V}{H}\left[\frac{b_1 l^2}{2H} + \frac{b_2 l^3}{3H^2} + \ldots + \frac{b_n l^{n+1}}{(n+1)H^n}\right]_0^h$$

For total volume, h = H, and as $\sum_{i=1}^{n} \frac{b_i}{(i+1)} = 1$

$$\text{Total volume} = \frac{V}{H}\left\{\left[\frac{b_1}{2} + \frac{b_2}{3} + \ldots + \frac{b_n}{(n+1)}\right] H\right\}$$

$$= V$$

Imposing the conditions on (8) and rearranging to obtain the equation in a linear form is straightforward and results in equation (11) which can be solved using a conditioned stepwise regression procedure.

$$\frac{d^2KH}{V} - \frac{2l}{H} = b'_2\left[3\left(\frac{l}{H}\right)^2 - \frac{2l}{H}\right] + b'_3\left[4\left(\frac{l}{H}\right)^3 - \frac{2l}{H}\right] + \ldots$$

$$+ b'_n\left[(n+1)\left(\frac{l}{H}\right)^n - \frac{2l}{H}\right] \qquad (11)$$

The coefficients of (10) can be readily obtained for

$$b_1 = 2\,(1 - \sum_{i=2}^{n} b'_i$$

$$b_2 = 3\,b'_2$$

$$\vdots \qquad \vdots$$

$$b_n = (n+1)b'_n$$

Equation (9) rearranged in linear form is very similar, the independent variables being the same, the dependent variable being

$$\frac{d^2K}{V} - \frac{2l}{H^2}$$

Results of the New Theory

Equation (11) was fitted to 2250 sets of observations in young crop radiata pine using the volume equation (4) and estimating 4 coefficients to give a 5th degree polynomial. All the coefficients were significant at the 95% level.

$$d^2 = \left[22.686\left(\frac{l}{H}\right)^5 - 44.310\left(\frac{l}{H}\right)^4 + 26.708\left(\frac{l}{H}\right)^3 - 3.5452\left(\frac{l}{H}\right)^2 + 1.1714\,\frac{l}{H}\right] \times (0.33021\,D^2 + 17.070) \qquad (12)$$

$SE(d^2) = 0.7792$ sq cm; $SE(d_e) = 1.43$ cm

The merchantable volume is then

$$V_m = V.\left[3.7810\left(\frac{l}{H}\right)^6 - 8.8621\left(\frac{l}{H}\right)^5 + 6.6771\left(\frac{l}{H}\right)^4 - 1.1817\left(\frac{l}{H}\right)^3 + 0.58571\left(\frac{l}{H}\right)^2\right] \qquad (13)$$

The curves for three trees are illustrated in Fig. 2.

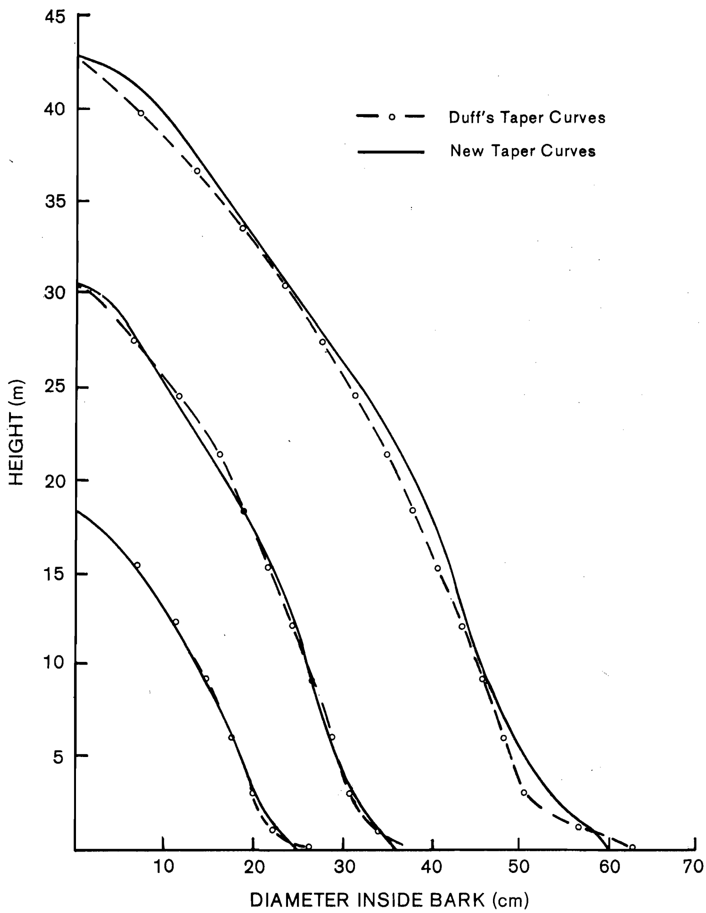

FIG. 2—Comparison of Duff's 1954 curves and new curves from equation (12) for young crop.

Neiloid, paraboloid and conoid sections are clearly visible. Because they are still widely used today, Duff's (1954) taper curves are plotted for comparison. For a given dbh and height they indicate less volume than the new curves. Apart from an indentation at about 3 m height, they are remarkably similar in shape. The new equation perhaps overpredicts diameter in the top 20% of larger trees, but only slightly. The average bias was calculated for 10% height intervals (as a % of total height) and is given below (bias = predicted — actual)

10% Height interval	95	85	75	65	55	45	35	25	15	5
Bias (cm)	+.10	+.10	+.04	+.02	+.04	+.02	+.01	+.01	.00	.00

When the term $\frac{d^2K}{V} - \frac{2l}{H^2}$ was used as the dependent variable, certain problems arose. The cubic term was found to be non-significant, but the next equation estimated without this term had all the terms significant.

$$d^2 = \left[0.23337 \left(\frac{l}{H}\right)^5 - 0.30488 \left(\frac{l}{H}\right)^4 + 0.14879 \left(\frac{l}{H}\right)^2 - 0.055030 \left(\frac{l}{H}\right) + \frac{2l}{H^2}\right] \times (0.33021\ D^2H + 17.070\ H) \quad (15)$$

This equation has a low SE(d_e), 1.40 cm, and has a similar shape to the curves of Fig. 2, except that it predicts negative values of d^2 for taller trees (40 m) in the region of the tip of the tree. This size of tree is at the edge of the data used to estimate the coefficients as the tallest young crop tree was 41 m. However, this feature was still highly undesirable. Calculating the best regression with one less term increased the SE(d_e) by 12% to 1.57 cm. These curves are illustrated in Fig. 3, the equation being,

$$d^2 = \left[0.0085000 \left(\frac{l}{H}\right)^5 + 0.040827 \left(\frac{l}{H}\right)^2 - 0.030056 \left(\frac{l}{H}\right) + \frac{2l}{H^2}\right] (0.33021\ D^2H + 17.070\ H) \quad (15)$$

DISCUSSION

For young crop radiata pine this equation was not as good as (12). Perhaps it would have been better to include higher order terms for independent variables as used by Bruce *et al.* (1968). However, the above discussion should illustrate some of the possible features of the equation. The same procedure was repeated for some 5360 sets of observations on old crop radiata pine, but no problems were encountered with either form of equation. Both had all terms to $(l/H)^5$ significant at 95% level and both were strictly non-negative for all tree sizes likely to be encountered now and in the future. Even so, the regression of the form of (8) had a smaller SE(d_e) = 2.44 cm than that of the form of (9) with a SE(d_e) = 2.69 cm. The maximum bias at the top 10% of the tree was 0.21 cm in equation (8), rapidly decreasing to negligible quantities over the lower portion of the tree. The alternative equation had even less bias, 0.14 cm at the top 10%.

It is felt that the new curves introduced above are a substantial improvement on the non-linear forms, as they more nearly represent tree shape. Their fit in the lower and middle regions of the tree is very good and the bias in the upper portion is very small. The SE(d_e) of 1.4 cm for young crop and 2.4 cm for old crop are comparable to those of the non-linear taper equations. They are calculated from raw data, unlike

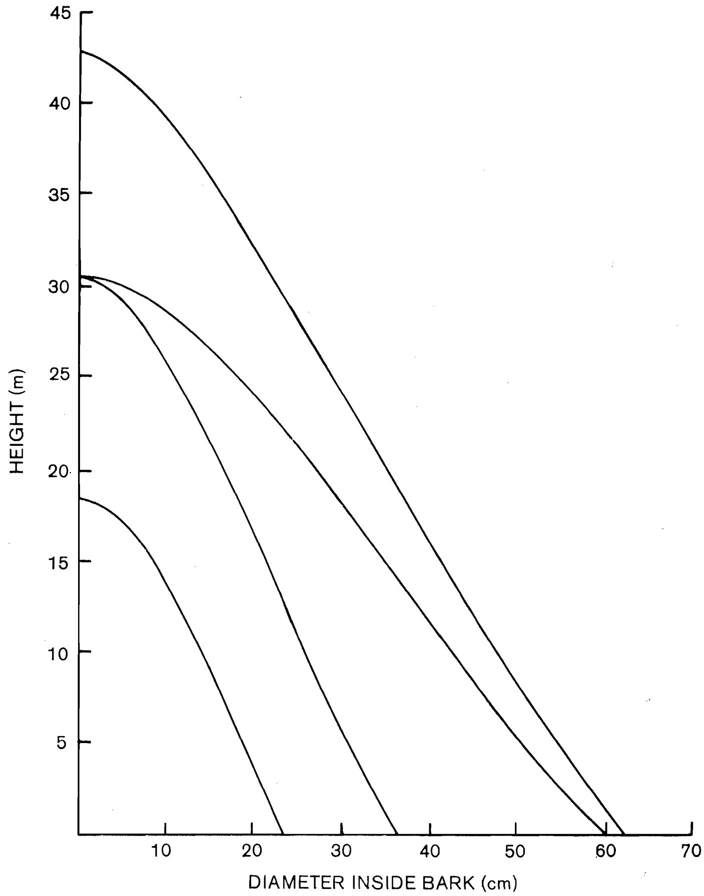

FIG. 3—Taper curves from equation (15) for young crop.

those calculated by Demaerschalk (1972a) which were calculated from the deviations of the calculated curves from hand-smoothed curves.

The calculated coefficients are for general volume and taper equations for Kaingaroa Forest. As such they represent the average shape of trees in Kaingaroa Forest and individual stands may differ from this average, but provided they are used in a context where the stand-to-stand variations average out they should be useful. The flexibility of the form of the equations is such, however, that they are ideally suited for experimental trials where the form of the trees from one treatment is likely to be different from that of another. In this case individual volume and taper equations should be estimated directly from the data of each treatment. Higher order polynomials could be tried if necessary; they are not used in this study as the shape and accuracy already obtained was adequate for the use to which they will be put.

CONCLUSION

The compatible taper equations appear to be very successful both in their freedom from undesirable characteristics and in their accuracy. They perhaps owe this success relative to other taper equations to the fact that incorporating a volume equation appears to determine the size of the tree, whilst the "free" coefficients of the taper equation can then estimate its shape. As Munro and Demaerschalk (1974) pointed out, the best fit is achieved for total volume, and then the fit for diameter is optimized. This should be appreciated by the forester whose major use for the equations will be for estimating total volume, followed by volumes in size assortments.

ACKNOWLEDGEMENTS

The authors wish to acknowledge the assistance of C. J. Mountford and W. Wooff, Foresters, Kaingaroa, for help in supplying the data and for useful comments during the course of the study.

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