VOLUME AND TAPER OF EUCALYPTUS REGNANS GROWN IN THE CENTRAL NORTH ISLAND OF NEW ZEALAND

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ABSTRACT

Tree volume and compatible taper equations have been developed for plantation **Eucalyptus regnans** F. Muell. growing in a central North Island forest. Previously published non-linear and polynomial forms of compatible taper equations were estimated but found to be inadequate for describing the shape of the whole stem. An extension of the non-linear form was developed, which characterised the neiloid, paraboloid, and conoid sections of the stem satisfactorily and for which the standard error of estimate of bole diameter is $\pm$ 13 mm. This development, it is claimed, goes some way to resolving the conflict between equation compatibility and prediction bias.

Keywords: Volume; taper; **E. regnans**.

INTRODUCTION

NZFP Forests Limited currently has 6352 ha planted in *Eucalyptus* species in the central North Island of New Zealand. *Eucalyptus regnans* accounts for 4060 ha of this total. The age-class distribution for this species is depicted in Fig. 1.

Stem volumes have been determined regularly in stands of *E. regnans* for the last 15 years or so, and by early 1986 a total of 364 trees had been destructively sampled and measured. Initial samples were obtained primarily from thinnings of young stands, but latterly the tree size range has been extended, using data from clearfell logging trials and various selective samples. The dbh frequency distribution of this database is illustrated in Fig. 2.

The measuring procedure involved taking diameters, both over and under bark, at 2-m intervals starting from breast height, until either the 10-cm or the 5-cm over-bark diameter was reached. Measurement was by callipers and the geometric mean of two right-angled diameter measurements was recorded. The under-bark measurements were obtained after stripping the bark. Stump diameter, stump height, and total tree height were also recorded and in the older age-class samples an intermediate measurement between the stump and breast height was made. Extensive graphical analysis was employed to identify and delete any obvious data measurement or recording errors.

Total tree volume, both over and under bark, was calculated by summation of the sectional volumes. The volume below the stump was represented by a cylindrical form, Smalian's formula was used for the intermediate section from the stump to the last recorded diameter, and the conic frustrum formula was used for the tip.

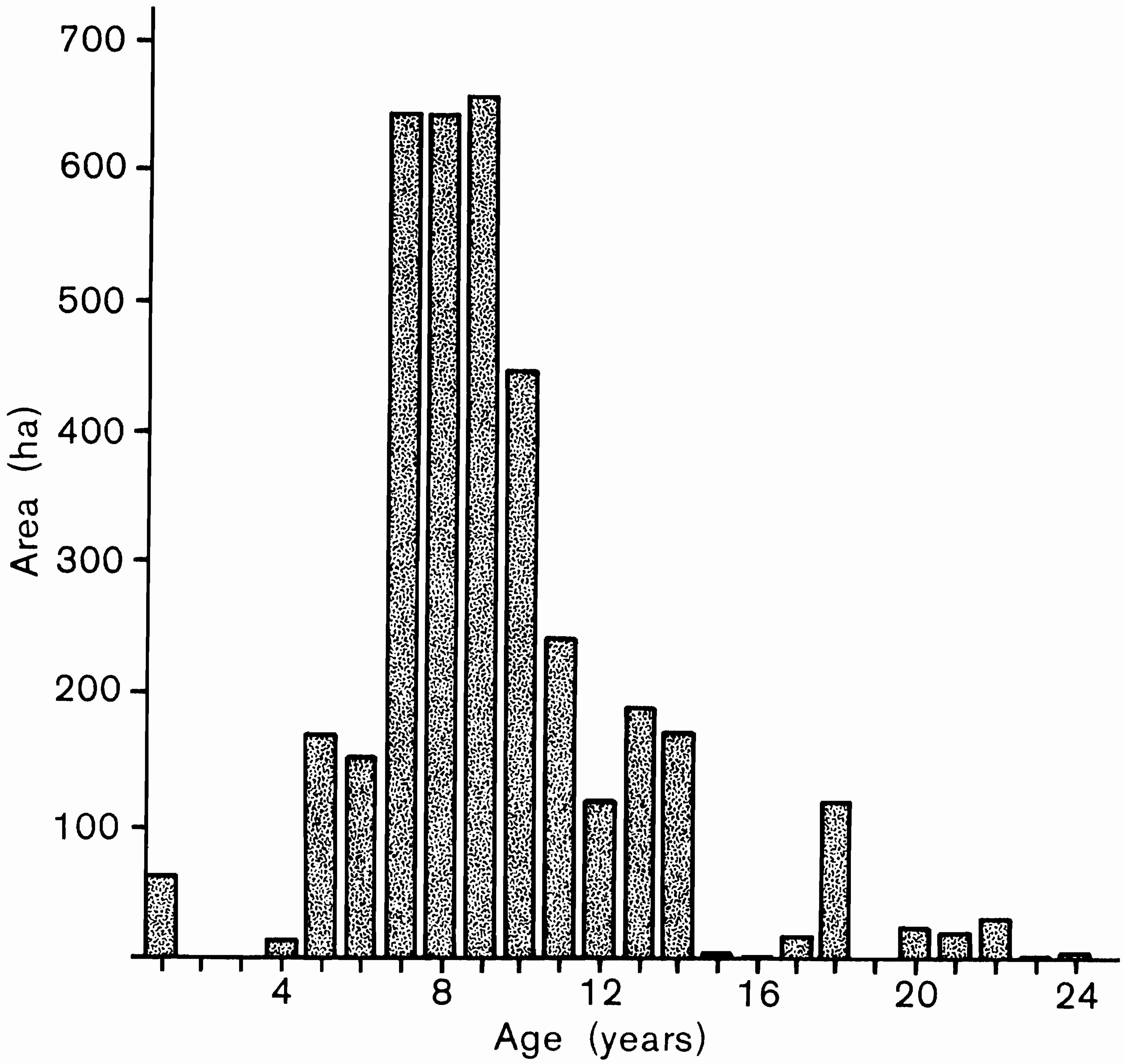

FIG. 1—**Eucalyptus regnans** age-class distribution.

VOLUME ESTIMATION

Various models of functional form:

$v = f(d,h,d^2h)$ were fitted to the data,

where:

v = underbark volume (m^3)

d = diameter at breast height (overbark) (cm)

h = total tree height (m)

The resulting graphs of residuals against the independent variables from the fitted models revealed that the observations had unequal variances, hence it was appropriate to apply weighted least squares. The problems associated with estimating regression coefficients when the assumption of homogeneity of variance has been violated, have been well documented (Furnival 1961; Cunia 1964; Honer 1965; Draper & Smith 1981). To determine the variance pattern and derive appropriate weights to apply, the

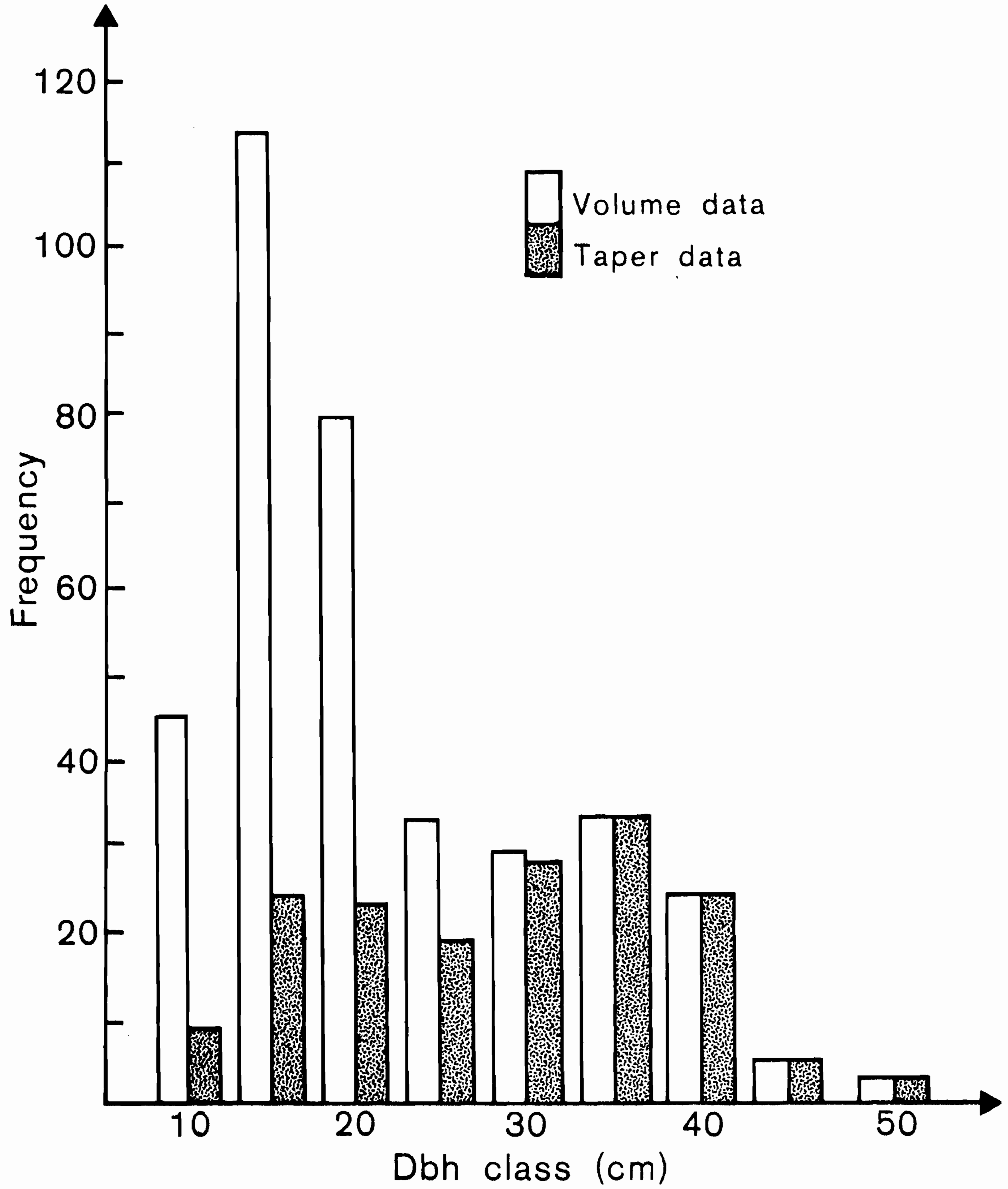

FIG. 2—**Eucalyptus regnans** database.

independent variable (d^2h) was subdivided into 10 ordered classes of an equal number of observations. The standard deviation of the residuals was then calculated for each group and found to be linearly related to the group mean values, i.e., the variance of the predicted volumes was found to be proportional to $(d^2h)^2$.

The reciprocal of $(d^2h)^2$ then formed the diagonal elements of the weighting matrix used in the subsequent regression analysis.

The constant form factor model (Spurr 1952) with weights equal to $1/(d^2h)^2$ was then fitted, giving the weighted least squares prediction equation:

$$v = 0.00002984 \ d^2h \qquad (1)$$

with standard error of the estimate $= 3.4 \ d^2h.10^{-6}$

Several other commonly reported volume functions (Clutter *et al.* 1983) were also fitted, none having smaller residual sums of squares or improved residual pattern.

For the evaluation of volume function precision, Honer (1965) requires that "volume errors should be predictable, relatively independent of tree size and average to within 10% of the size class means in at least 95% of the classes having more than 5 observations". The volume errors are given in Table 1 as a percentage difference by dbh class; the volume function is acceptable in terms of the above definition.

The over-all percentage difference was −1·7 on 364 observations, where percentage difference is defined as:

100 (observed volume − predicted volume) / observed volume.

TABLE 1—Volume equation percentage errors by dbh class

Diameter class (cm)	Actual volume (m^3)	Estimated volume (m^3)	Volume difference (%)	Number of observations
9	0.031	0.033	−6.5	5
11	0.054	0.057	−5.6	24
13	0.079	0.078	1.3	43
15	0.132	0.126	4.5	51
17	0.165	0.159	3.6	42
19	0.214	0.205	4.2	33
21	0.259	0.263	−1.5	35
23	0.326	0.334	−2.5	18
25	0.537	0.525	2.2	12
27	0.515	0.509	1.2	11
29	0.678	0.691	−1.9	16
31	0.690	0.744	−7.8	9
33	0.808	0.874	−8.2	8
35	1.122	1.139	−1.5	15
37	1.332	1.282	3.8	16
39	1.372	1.410	−2.8	13
41	1.628	1.659	−1.9	6

(For classes with five or more observations)

TAPER ESTIMATION

Unfortunately, documentation of tree volume sampling prior to 1980 was restricted to volume summaries only. Consequently, the dataset used in the formulation of the taper equation was a subset of the volume data, being restricted to samples undertaken after 1980. In all, 2208 observations on 177 trees were recorded and the dbh frequency distribution is illustrated in Fig. 2. These measurements were then partitioned into lower bole, mid bole and upper bole groups. For each tree not more than one observation was then randomly chosen from each of these groups, yielding up to three independently chosen diameters per tree and 474 observations in total.

In describing the various taper functions, the following notation will be used:

Let v = total underbark volume (m^3)
d = diameter at breast height (overbark) (cm)
h = total tree height (m)
l = distance from the tip of the tree (m)
d' = diameter underbark at l metres from the tip (cm)
v_l = underbark volume between the top of the tree to the point l.
$K = \pi/40\,000$.

The approach used in estimating the taper equation has been defined as a volume-based compatible estimation system (Munro & Demaerschalk 1974). Firstly, a volume equation is estimated from volume data and then a taper equation is formulated from taper data such that the two systems of equations yield identical total tree volume, i.e., compatibility with respect to volume is achieved.

The compatible non-linear model:

$$d'^2 = \frac{v}{K.h}(r + 1)\,(l/h)^r \qquad (2)$$

where r is a parameter to be estimated, and the polynomial model:

$$d'^2 = \frac{v}{K.h}\left(\beta_1(l/h) + \beta_2(l/h)^2 + \ldots + \beta_n(l/h)^n\right) \qquad (3)$$

where $\beta_1, \beta_2 \ldots, \beta_n$ are regression parameters with the restriction that $\sum_{i=1}^{n} \frac{\beta_i}{i+1} = 1$, were both considered and coefficients estimated as per Goulding & Murray (1976). Model 2 was rejected immediately as it failed completely in describing the butt swell and the top section of the tree. Model 3, with $n = 5$, adequately described the lower bole section but was found to over-estimate diameter consistently in the upper bole range. The same trend was also noticeable in the bias table of Goulding & Murray (1976), with Model 3 fitted to *Pinus radiata* D. Don data.

A variant of the above non-linear model was then proposed in which the exponent r was replaced by a function of (l/h). That is,

$$r = f(l/h).$$

To retain the desirable compatibility property (*see* Appendix) the following taper model was proposed:

$$d'^2 = \frac{v}{K.h}\left(f(l/h)\,/\,(l/h) + f'(l/h)\,.\,\ln(l/h)\right)(l/h)^{f(l/h)} \qquad (4)$$

and $f(l/h)$ must be differentiable on the interval [0,1]. In fact, the database reflects a range of (l/h) from 0.11 to 1.0, that is, from ground level to approximately 90% of tree height, well above the reasonable limit of merchantability. For this reason, the model being undefined at the tip, is not seen as a disadvantage.

Model 4 was fitted to the taper dataset by non-linear least-squares. The following prediction equation for diameter under-bark was thus produced:

$$d'^2 = \frac{v}{K.h} (f(x) / x + f'(x) \,.\, \ln(x))\, x^{f(x)}$$

where $x = (l/h)$

and $f(x) = 0.7415 + 8.2985\, x - 13.1251\, x^2 + 6.7841\, x^3$

All parameter estimates were significantly different from zero at the 5% significance level and the standard error of the estimate of bole diameter was calculated to be ± 13 mm from the formula

$$SE_E = [\Sigma(d - \hat{d})^2 / (n - m - 1)]^{\frac{1}{2}}$$

where d and $\hat{d}$ are the actual and predicted diameters respectively,
n is the number of observations, and
m is the number of parameters in the regression model (Demaerschalk 1972).

MODEL GOODNESS-OF-FIT AND VALIDATION

The results of an analysis of percentage diameter bias for the polynomial Model 3 and for Model 4 are given in Table 2. Percentage under-bark diameter bias has been calculated by 5% height classes on the full dataset (2208 observations on 177 trees), and is defined as:

100 (observed diameter – predicted diameter) / observed diameter.

TABLE 2—Taper function percentage diameter bias (for Taper Models 3 and 4)

Height percentile	Diameter bias (%)		Number of observations
	Model 4	Model 3	
0– 5	0.8	0.5	240
6–10	1.2	0.9	174
11–15	0.6	–0.2	122
16–20	0.9	0.3	129
21–25	0.3	–0.2	122
26–30	0.3	–0.4	123
31–35	–0.5	-1.3	111
36–40	–0.7	-1.7	118
41–45	0.3	-1.2	119
46–50	–0.3	–2.6	126
51–55	1.7	–1.0	120
56–60	3.0	–0.6	108
61–65	2.3	–2.3	130
66–70	1.6	–3.9	113
71–75	0.3	–7.8	121
76–80	–3.8	–16.3	99
80–85	–11.7	–36.8	80
86–90	–6.7	–58.4	47

The overall average bias was 0.4% for Model 4, and 4.5% for Model 3, for all 2208 observations. The apparent overprediction by Model 4, of diameter above 75% of tree height, is still more acceptable than the bias displayed by the polynomial model and points to this section of the tree being relatively narrow with little taper. This characteristic is further reinforced when it is calculated from the Model 4 taper equation that 95% of total tree under-bark volume lies below the 75% total height mark, and approximately 82% of total volume lies below half height.

The typical tree profile, as predicted by Model 4, for a tree of 30 cm dbh and 30 m height is illustrated in Fig. 3. Two points of inflexion are defined for the tree

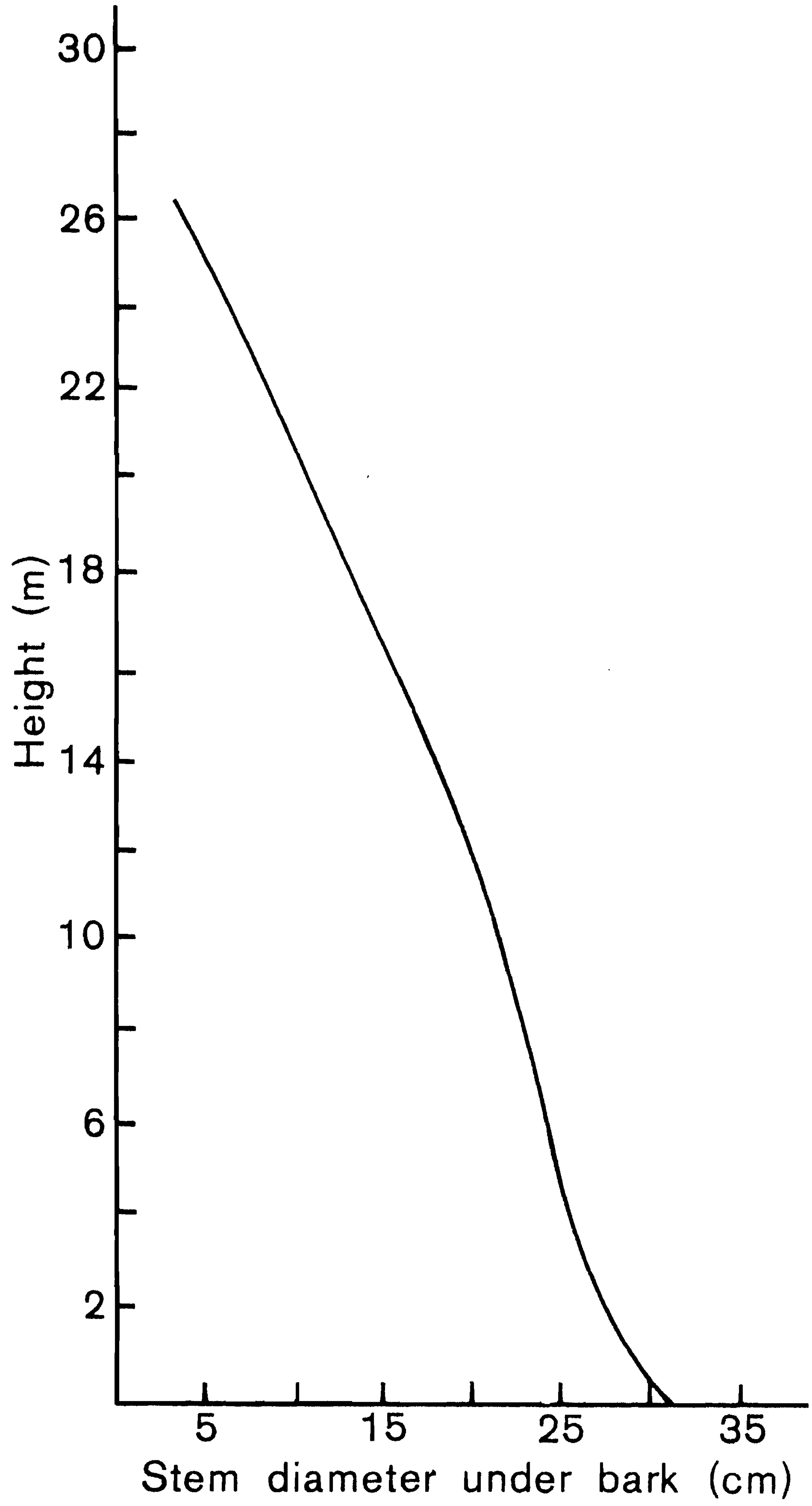

FIG. 3—*Eucalyptus regnans* tree profile - model defined only to 90% of tree height.

profile – one at approximately one-quarter total height, the other at approximately one-half height, reflecting transitions between the butt neiloid section to the mid paraboloid to the upper conic section respectively.

The result of an analysis of percentage height bias by 5% height class is given in Table 3. A random sample was selected from the full dataset, yielding 445 sample points, stratified by height. Using the known diameter under-bark (d′), the height at which this diameter occurs (h′), and the tree dbh and height, Newton's iterative technique (Conte & de Boor 1972) was used to determine $\hat{h}'$, the estimate of h′, using the Model 4 taper equation. Percentage height bias is defined as:
100 (observed height – predicted height) / observed height.
The over-all height bias was 0.8% measured on 445 sample points.

TABLE 3—Taper function percentage height bias (Taper Model 4)

Height percentile	Height bias (%)	Number of observations
0– 5	5.9	30
6–10	-2.1	19
11–15	5.9	25
16–20	3.9	23
21–25	3.2	32
26–30	6.5	22
31–35	-0.7	26
36–40	-1.7	25
41–45	3.3	26
46–50	-0.6	25
51–55	1.9	21
56–60	2.6	28
61–65	0.1	19
66–70	1.8	22
71–75	1.3	27
76–80	-0.3	23
81–85	-2.0	29
86–90	-1.1	16

Individual tree volume and taper data for a further 22 trees were made available by the Forest Research Institute Management of Eucalypts Co-operative and, as these data represented an independently collected source of information from the central North Island, a validation exercise was undertaken. Eleven of the trees were sampled from Cpt 83, Rotoehu Forest, planted in 1961 and measured in 1978. The other tree information came from 1976 plantings at Hamurana, Rotorua, measured in March 1987. Both samples were of *E. regnans* and the pooled sample displayed a dbh range of 13.9 to 44.7 cm and a height range of 14.4 to 32.5 m.

The only significant difference between the measurement method described previously and the method used for the above samples was the distance used between successive diameter measurements. The sample from Hamurana used a standard 3-m interval, whereas a variable length interval, not greater than 3 m, was used for the Rotorua sample.

Individual tree under-bark volume was calculated as previously described, and an estimated volume calculated using Equation 1. A paired t-test could not reject the hypothesis that the mean difference between these two volume estimates was equal to zero.

The results of an analysis of percentage diameter bias for taper Model 4 are summarised in Table 4. Percentage under-bark diameter bias, as previously defined, has been calculated by 5% height classes for the available 182 observations; however, classes with less than four observations have been deleted. Taper Model 4 provides a satisfactory fit for this validation dataset, displaying a total bias of 1.9%.

TABLE 4—Taper function percentage diameter bias (Taper Model 4 — FRI validation data)

Height percentile	Diameter bias (%)	Number of observations
0– 5	5.0	37
6–10	3.9	20
11–15	4.7	10
16–20	3.3	12
21–25	2.3	8
26–30	–0.6	11
31–35	2.6	11
36–40	2.1	8
41–45	0.1	10
46–50	2.9	6
51–55	–0.3	10
56–60	–1.5	9
61–65	9.5	12
66–70	–8.9	7
71–75	–6.6	4
76–80	6.0	5

DISCUSSION

The flexibility of taper Model 4 is demonstrated by considering the expression:

$$d'^2 = p\ x^r$$

where d' = underbark diameter of the stem at x,

x = distance measured from the tip,

p and r are constants.

This is the general form for the curve that defines various geometrical solids, when the stem is considered as a solid of revolution (Grosenbaugh 1966). The constant r defines the geometric shape of the curve.

Model 2 was introduced as a volume compatible taper equation and was derived directly from the above general form, i.e.,

$$d'^2 = p\, x^r \qquad (2a)$$

where $p = \dfrac{v}{K.h}(r + 1)$

$x = (l/h)$

By defining Model 4 as:

$$d'^2 = p\, x^{f(x)} \qquad (4a)$$

where $x = (l/h)$

$f(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$

and $p = \dfrac{v}{K.h}\,[f(x) / x + f'(x) \,.\, \ln(x)]$,

the shape parameter, r, has been replaced by an expression, f(x), which is allowed to vary continuously over the length of the stem, permitting a more precise definition of the entire bole shape to be made. In defining the taper of *E. regnans,* f(x) in the above taper model was best estimated as a cubic polynomial. Other functions are, however, possible provided that the first derivative exists over the domain of x and allowance is made for different species.

CONCLUSION

In this paper a compatible system has been described which is largely free of bias (Tables 1, 2, and 3) and conforms with most of the desirable properties of volume and taper equations (Goulding & Murray 1976). Compatibility and estimates free of bias are difficult to attain. Demaerschalk & Kozak (1977) admitted to the difficulty of formulating an unbiased compatible system of volume and taper functions when using one mathematical model to describe the whole-tree profile. Cao *et al.* (1980) concluded that some compatible volume and taper models apparently sacrifice precision in diameter estimation to ensure the compatibility of the taper model. However, the volume-based approach used in this paper, coupled with an extremely flexible taper equation, progress greatly toward the reconciliation of the compatibility *versus* bias conflict.

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APPENDIX

The volume compatibility is demonstrated as follows:

$$v_h = \frac{v}{h}\int_0^h [\, f\,(l/h)/(l/h) + f'\,(l/h)\,.\,\ln(l/h)\,]\,(l/h)^{f(l/h)}\,dl$$

set x $= l/h \quad dx = (1/h)\,dl$

then $v_h = v\int_0^1 [\, f\,(x)/x + f'\,(x)\,.\,\ln(x)\,]\,x^{f(x)}\,dx$

set z $= x^{f(x)}, \; dz = (f\,(x)/x + f'\,(x)\,.\,\ln(x))\,x^{f(x)}\,dx$

i.e., $v_h = v\int_0^1 dz$

$= v\,[\,x^{f(x)}\,]_0^1$

$v_h = v$

With $x = (l/h)$, the first derivative of the taper equation is defined as:

$$\frac{d\,(d^2)}{dx} = \frac{v}{K.\,h}\,[\,2f'\,(x)/x - f\,(x)/x^2 + f''\,(x)\,.\,\ln(x) + (f\,(x)/x + f'\,(x)\,.\,\ln(x))^2]\;x^{f(x)}$$

and an iterative method such as Newton's method can be used to calculate the height of any given under-bark diameter.