

CUBIC SPLINE CURVES AND CALCULATION OF VOLUME OF SECTIONALLY MEASURED TREES

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ABSTRACT

When stem volume is estimated from measurements of cross-sectional area using Smalian's composite formula and data from *Pinus radiata* in New Zealand errors in volume rapidly increase as intensity of measurement decreases, exceeding 8% when the interval between measurements is greater than 5 m. Newton's composite formula or Romberg's method can halve this error but are applicable only when the intervals between measurements are uniform. Integrating a cubic spline curve fitted through the data points can estimate volume with only 60% of the error of Smalian's formula. This function is suited for use with dendrometer measurements of standing trees and, provided that the distances between measurements are less than 5 m, errors can be confined to less than 5% of the volume.

INTRODUCTION

Where a tree or a log has been measured for diameter at intervals along its length, its volume has been estimated either from formulae or with graphical methods. Smalian's, Huber's and Newton's composite formulae are most commonly used while the frustum of a cone has been applied to a lesser degree. All of these are well described in mensuration text books (see Husch *et al.*, 1972). Newton's formula is generally regarded as the most accurate followed by Huber's.

Where trees have been measured with irregular section lengths, as in the dendrometry of standing trees, Newton's and Huber's formulae cannot be applied, whereas Smalian's formula can be used. Graphical methods may be used to overcome the large errors that can occur, for example that by Reineke (1926) and the Taper Line method of Gray (1956).

Carron and McIntyre (1959) found errors of -2% when Huber's formula using 3 m (10 ft) sections was compared to a standard volume using 0.3 to 0.45 m (1-1½ ft) sections in *Pinus radiata* D. Don. Young *et al.* (1967) using water displacement as a standard obtained average errors of -3.6% to -5.1% for Huber's and $+8.7\%$ to $+10.0\%$ for Smalian's on 4.9 m (16 ft) logs, with maximum individual errors of $+44\%$ for Smalian's and $+15\%$ for Huber's method (all species). On 1.2 m (4 ft) logs, average errors were -3.5% and -4.6% for Huber's and Smalian's formulae respectively.

Dargavel and Ditchbourne (1971), also using water displacement, found that Smalian's formula with measurement intervals of 0.9 m and 1.6 m gave unbiased

estimates of the volume of small *P. radiata* trees.

Ellis and Duff (1973) reported on a comprehensive study involving 450 trees in New Zealand. They used trees measured every 0.3 m (1 ft) as their standard and found that Newton's and Huber's formulae gave errors which increased gradually from +0.2% and -0.4% with 1.2 m (4 ft) sections to 3.0% and -2.9% with 6.1 m (20 ft) sections. Smalian's on the other hand gave errors which increased rapidly from 1.5% with 1.2 m sections to 6.1% with 3 m sections and then to 14.7% with 6.1 m sections.

Theory

A stem of a tree has been thought of as being formed from a composite of geometrical solids; a neiloid frustum at the butt, a paraboloid frustum in the centre and a cone at the tip. Grosenbaugh (1966) noted that these shapes are obtained by rotating around the x axis a curve of the general form

$$y^2 = px^r$$

where y is the radius of the stem

x is the distance from the tip

p and r are constants, the latter defining the shape of the solid.

The volume can thus be found from the resulting solid of revolution. However, this is not a precise definition of the problem of calculating volume, as the point where one shape gives way to another remains undefined. Moreover, as foresters are only too well aware, a perfectly defined mathematical shape does not approach the reality of a rough log, measured with some error at infrequent intervals along its length, sometimes under trying conditions. A more realistic definition is as follows.

The problem of mathematically defining the shape of the stem with a view to calculating its volume is that of finding a "smooth" function of cross-sectional area in terms of length (l) along the stem, where the true form of the function $f(l)$ is unknown but where cross-sectional area can be evaluated at given points on the stem (albeit with some measurement error). This smooth function can then be integrated to give the volume of the stem, interpolated to derive diameters between measurements, or differentiated to give rates of taper at given points.

Numerical Quadrature enables the integral of a function to be estimated and an application to trees was recognised by Reineke in 1926 with his graphical method. However, since the advent of the electronic computer, methods of numerical analysis have been vastly improved.

Smalian's and Newton's composite formulae are equivalent to the Trapezoidal rule and Simpson's rule respectively, which belong to a family of estimators known as the Newton-Cotes formulae. Newton's formula owes its success over Smalian's because it approximates each segment of $f(l)$ by a second order polynomial, whilst Smalian's formula is a simple line segment. There are two sources of error in numerical quadrature methods; truncation error caused by using a simple approximating formula when a more complex one is better, and "noise" caused by errors in the data, round-off error, etc. A further method, termed Romberg's method, uses an estimate of the truncation error in the Newton-Cotes formulae to provide a better estimate of the integral. This method has been applied to log volume calculations by Goulding (1971) and is described in most texts on Numerical Analysis, e.g. Wilf (1967).

To show the order of error between the three methods, the integral $\int_0^{\pi/2} \sin x \, dx$ was estimated using 8 sections by Smalian's rule, Newton's rule and Romberg's method, Table 1.

TABLE 1—The integral of $\sin x$ from 0 to $\pi/2$

	Value	Error
Smalian's Rule	0.996786	0.003214
Newton's Rule	1.000134	0.000134
Romberg's Method	1.000001	0.000001

Romberg's method has the limitations that the number of sections used must be a power of 2 and that the section lengths must be equal. This does not preclude its use several times over parts of the stem, for example the volume of a tree 42 m long measured every 4 m could be evaluated by Romberg's method for the first 8 sections and again for the next 2 but using Smalian's for the last. This is how Romberg's method is applied in the remainder of the paper to estimate the volume of trees with a fixed number of measurements.

The Natural Cubic Spline Curve

The difficulty imposed when trees have been measured at irregular intervals along the stem, e.g. by optical dendrometer, can be avoided by the use of a natural cubic spline curve, and the large errors introduced by the use of Smalian's formula avoided to some extent. The use of a computer to perform the calculations is essential, but not only can the volume of the stem be obtained from the integral but the cross-sectional area and rate of taper at any point can be obtained as a by-product.

A spline is a long piece of thin wood or other flexible material used by draughtsmen to fair-in a smooth curve between specified points termed "knots". A mathematical spline curve approximates the draughtsmen's spline by a piecewise polynomial, usually a different polynomial for each arc between pairs of points. A cubic spline uses a cubic polynomial. Each pair of adjoining arcs about a data point fits the data point exactly and has the same value of the 1st and 2nd derivatives at that point. The resultant composite curve appears smooth to the observer.

Mathematically a natural cubic spline function $S(x)$ is characterised by three properties.

1. $S(x)$ is given in $[x_i, x_{i+1}]$, $x_1 < x_2 \dots \dots < x_n$ by some polynomial with a maximum degree of 3.
2. $S(x)$, its first and second derivatives are continuous on the interval $(-\infty, \infty)$.
3. $S(x)$ is a line segment in the intervals $(-\infty, x_1)$ and (x_n, ∞) .

Furthermore the function is "smooth" in the sense that $\int_{x_1}^{x_n} (S''(x))^2 \, dx$ is a minimum, where the double prime denotes the second derivative.

Cubic spline functions owe their popularity to their best approximating properties for either the interpolating value, the value of the 1st and 2nd derivatives at a given

point on the curve, or the integral over the interval. They are best in the sense that they can best approximate interpolated and integrated values of an arbitrary class of smooth functions though for a particular use a better approximation method may exist.

Other spline functions exist and are receiving wide application in the estimation of the form of both mathematical functions and solid objects. Wahba (1976) used "Histosplines" to fit a curve to a histogram and obtain an empirical probability density function; Clark (1976) uses "B-splines" to represent the surfaces of solid objects in a three-dimensional computer design system. Sommers (1976) noted the application of spline curves in Forest Meteorology.

In this application a natural cubic spline curve has been fitted to the measurements of the cross-sectional area at various points along the stem of the tree and the volume obtained from the integral of the curve. The method of calculation was that given by Greville (1967) who also gives a theoretical justification for the curves. Ahlberg *et al.* (1967) give a simpler algorithm in a general reference book on spline curves.

METHOD

To illustrate how beneficial the use of a natural spline curve could be in calculating volume a small study was carried out on a sample of trees from the study reported by Ellis and Duff (1973). Data from 48 *P. radiata* trees were selected, the distribution of heights and dbh's given below.

TABLE 2—Distribution of data

Total Height (m)	15	20	25	30	35	40	45	50
Frequency	7	7	8	7	4	5	6	4
Dbh (cm)	20	30	40	50	60	70		
Frequency	13	8	9	9	6	3		

The trees selected were limited to those of normal form, that is reasonably straight stems with single leaders. Diameter overbark had been measured at 0.3-m intervals along the stem, measuring on nodes and swellings if the measurement point occurred there. A caliper was used and only one measurement taken. The volume of each tree was calculated by Newton's and the Cubic Spline formulae using all measurements. The difference between the methods averaged less than 0.1% and it was concluded that the level of intensity of measurement was more than adequate to obtain a true estimate of volume. Any remaining errors would be due to "noise", that is measurement errors, eccentricity in the cross-section, round-off errors, etc., and would not be removed by using a different formula or increasing the intensity of measurement. As there had been no attempt to avoid "unrepresentative measurements" the volume calculated was an estimate of the true volume of the tree.

To investigate the effects of the method of calculation and the intensity of measurement the volumes of the trees were then recalculated for varying distances between measurements in four ways.

1. Smalian's formula $\frac{(A_i + A_{i+1})}{2} L$
2. Spline Curve Integration
3. Newton's formula $\frac{(A_i + 4A_{i+1} + A_{i+2})}{3} L$
4. Romberg's method

where A_i = Cross-sectional area over bark at *i*th point of measurement

L = distance between measurements.

Volumes were calculated for each method for varying intensities of measurement, using every 2nd, 4th, 6th . . . 20th measurement and the errors obtained as a percentage of the volume calculated from using all measurements. In addition, the traditional New Zealand Forest Service method had been used and its average error obtained. Here, diameters were measured from the butt every 3 m (10 feet) for trees taller than 17 m, every 1.5 m for trees shorter, with additional measurements at 0.15, 0.75 m and 1.5 m; Smalian's formula was used for all sections except the top where a conic formula was applied. A final method tested was the Taper Step method as specified by Whyte (1971; p. 77). Diameters were measured at 0.6 m and 1.4 m above ground and then next diameters taken to given taper steps of 2.5 cm if the last diameter was less than 25 cm, 5 cm if the last diameter was greater than 25 cm. The volume of each section was calculated using the Conic formula:

$$(A_i + A_{i+1} + \sqrt{A_i A_{i+1}}) \cdot L/3$$

By and large this procedure ensured that internodal diameters were used, and would by no means be far from reality when compared to the field procedure implemented in the Forest Service.

The average number of measurements used in each method and intensity was also calculated.

RESULTS

In Fig. 1 the average absolute percentage errors were plotted against the distance between measurements. Except for the Forest Service and Taper Step methods which underpredicted, all other errors were overpredictions.

The Forest Service method had a percentage error of $-0.3\% \pm 0.57\%$ at 95% confidence level with an average number of measurements of 13.5. The Taper Step method as specified by Whyte had a percentage error of $-3.46\% \pm 0.75\%$ for 12.5 measurements. By averaging the number of measurements used for each intermeasurement distance, an expression of the work content at each level of accuracy was obtained and illustrated in Fig. 2.

As can be seen, where the distance between measurements is less than 2 m, all four methods have errors less than 2%. However, increasing the interval between measurements rapidly increases the percentage error of volume calculated by Smalian's method, so that 6 m intervals have errors of about 10%. This is comparable with the results in

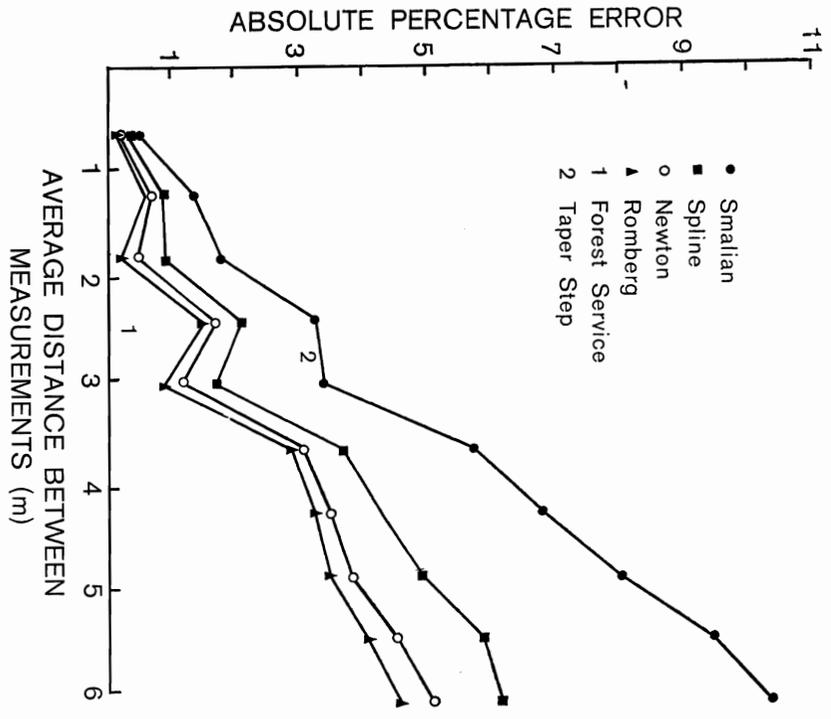


FIG. 1—The error in the estimate of volume for the various methods.

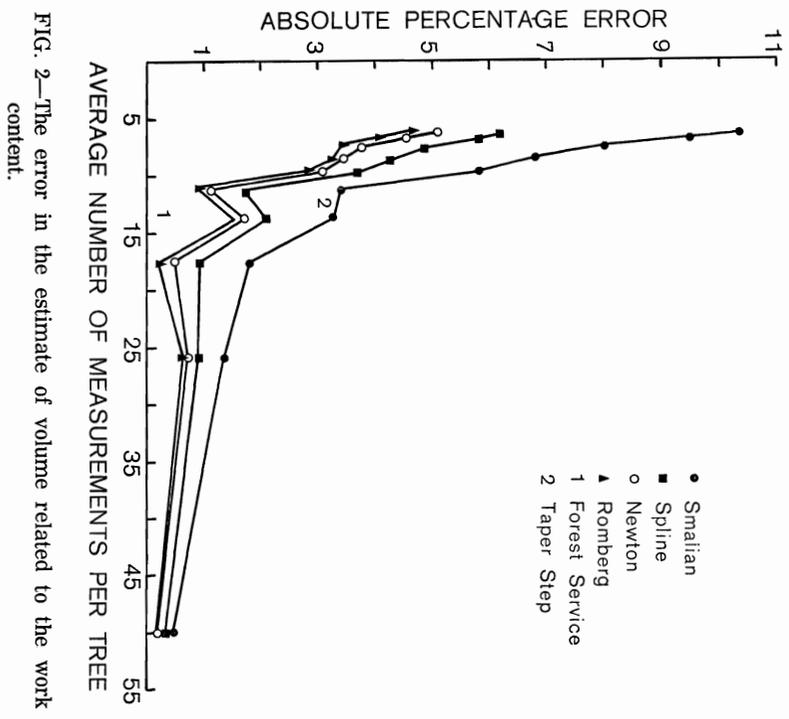


FIG. 2—The error in the estimate of volume related to the work content.

the literature given in the Introduction.

The spline curve volume error is on average about 60% of the equivalent error when using Smalian's formula. Even where the section length is 5 m, the error is only about 5%. Still less error can be obtained using the composite Newton's formula and Romberg's method, the latter having errors only 85% as large as those when using Newton's formula, which has errors only 50% the size of the equivalent error of Smalian's formula. Where sectional measurements are taken less than 2 m apart, the errors involved in the spline curve, Newton's formula and Romberg's method would be due to "noise" rather than truncation error due to an inadequate approximating formula or to an insufficient number of measurements when taken at strictly equal intervals. This implies that no improvement would be obtained with a different formula or with a higher intensity of measurement.

None of these four comparisons takes into account varying the location of the measurements. The Forest Service method concentrates extra measurements at the butt of the tree, whilst the Taper Step method has the frequency of measurements proportional to the taper of the tree. The error of between -0.9% and $+0.3\%$ for the traditional Forest Service method was the smallest error of all procedures at an equivalent intensity of measurement. The Taper Step method, by concentrating at mid-internodes and using a conic formula, has an error between -4.1% and -2.7% , comparable to that when using Smalian's formula. The equivalent errors of the Spline Curve, Newton's and Romberg's method were 1.8%, 1.4%, and $1.2\% \pm 0.8\%$ respectively, assuming that the same number of measurements was used.

The variances of the percentage error were also calculated and plotted (Fig. 3). The variances of the errors in the spline curve, Newton's formula and Romberg's method were almost identical, and only those of Newton's formula have been plotted for clarity.

The variance of Smalian's method was higher than the other three methods, indicating that individual tree estimates of volume were likely to be less reliable than the other methods. There is no theoretical justification for the Taper Step method to have a lower variance of errors than the other methods, and as can be seen it is comparable to that obtained using Newton's formula, though lower than that of Smalian's.

DISCUSSION

The results indicate that Newton's and Romberg's method calculate volumes with smallest errors for trees measured at regular intervals. However, by concentrating measurements at the butt section the New Zealand Forest Service method outperforms all methods at the same level of intensity of measurement, despite using Smalian's formula. It achieves such a low level of error that improving the formula would not give a perceptible increase in accuracy. By increasing the section length of the Forest Service method and using Newton's or Romberg's method to calculate the volumes, the work content could be reduced for small expense in accuracy. The Taper Step method performs poorly, firstly by taking measurements at mid-internodes to give a biased underestimate of tree volume, and secondly by concentrating measurements at the top, due to changing the taper step size when the large-end diameter decreased to below 25 cm. It was estimated that 50% of the measurements were spent calculating less

than 5% of the total volume and confirms practical experience. A modification of the taper step method is useful in badly malformed stems where fixed distances between measurements can be difficult to implement. Where a sharp reduction in diameter occurs the stem should be measured directly above and below the sudden taper. Otherwise measurements should be taken at stump height, half breast height, breast height and about 3 m. Over the remainder of the stem with a gradual taper, measurements should be made at regular intervals corresponding to the level of accuracy required. If the true volume is required, then measurements must not be confined to the mid-internodes, though points with a severe abnormality in cross-section should be avoided, for example, deep scars and local large swellings.

The cubic spline curve estimates volume with more accuracy and greater precision than Smalian's formula. It can provide a better model to interpolate diameters than the assumption that the segments between measurements are frusta of cones (diameter linearly related to length) or frusta of parabolas (cross-sectional area linearly related to length). Figure 4 illustrates the profile of a tree 23.5 m tall, with a d.b.h. 33 cm and cross-sectional areas measured every 0.3 m and the estimated profile based on a cubic spline curve fitted to 12 arbitrarily selected points ("knots"). The sinuous nature of the spline curve can be seen, especially that induced by the measurements at the base of the tree and at a measurement at 12 m on one of the nodes.

These tests of the spline curve have been based on normal trees where a smooth stem profile is likely to be applicable. In practice it has been found that some combinations of malformation and location of measurements cause the spline curve to estimate interpolated diameters less than satisfactorily over parts of the stem. For example when long and short intervals between measurements are side by side, oscillations in the interpolated curve can occur, as in Figure 4 at the butt of the tree, where "knots" have been taken at 0.0 m, 0.3 m, 1.4 m and 3.0 m. This occurs infrequently and is due to the requirement that the curve be "smooth". With some heavily malformed trees, a form with a discontinuous first derivative approximated by piecewise line segments may be more appropriate. Alternatively separate spline curves can be fitted to the data on either side of the malformed section and the first derivatives estimated from the spline curves at the junctions. A further spline curve can be fitted over the malformed section using these first derivatives at the end points. The resultant composite curve would have a continuous first derivative over the whole tree, but have discontinuities (usually large) in the second derivative at the junction points.

A program has been written to calculate the total volume of a tree (inside and outside bark) using a natural cubic spline curve (Deadman, 1975). In addition to calculating total volume, the spline curve is used to provide an interpolation technique to estimate volumes to given lengths or merchantable diameters not already measured, subject to any user-specified conditions. For example, if the user specified L0.5; DI250; (L0.5; DI250; >4.0) the program would calculate the volume above a stump height of 0.5 m to an inside bark diameter of 25 cm provided the log length between stump height and diameter inside bark of 25 cm was greater than 4 m.

Though not considered in this study, a point mentioned by Ellis and Duff (1973) is worthy of note. With long distances between measurements, Huber's formula is more robust than other formulae. They point out that even with an interval of 10 m or more,

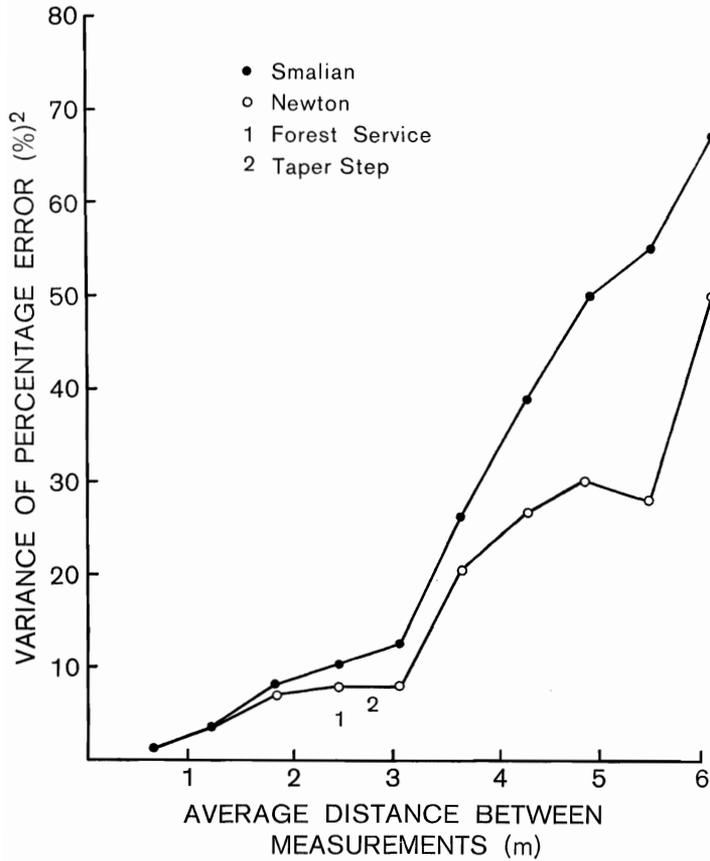


FIG. 3 (above)—The variance of the percentage error.

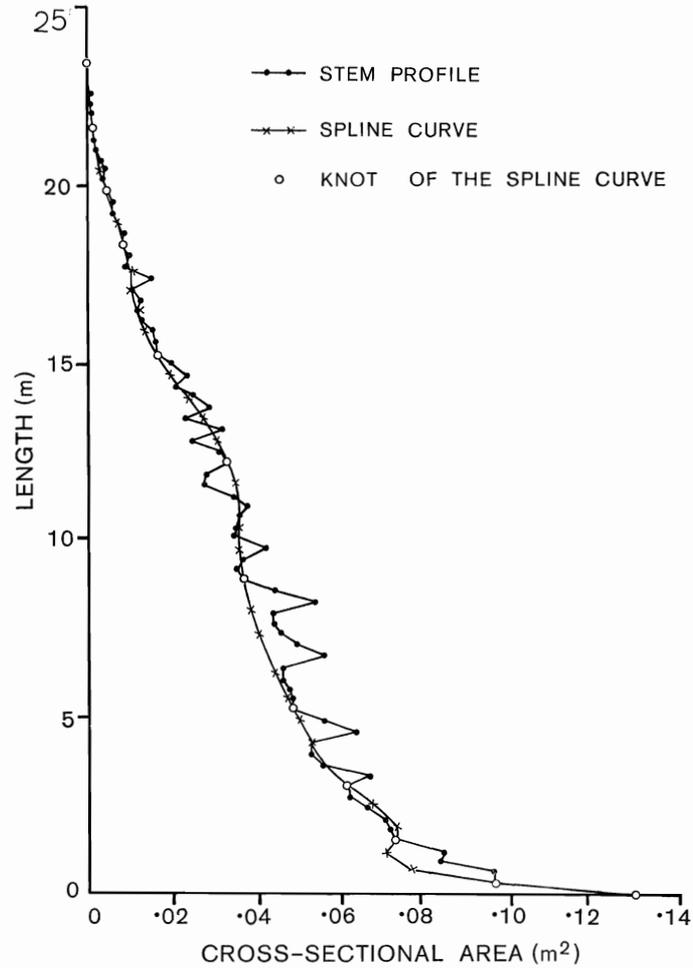


FIG. 4 (right)—The profile of a tree 23.5 m tall with a d.b.h. 33 cms.

the underprediction in Huber's formula was only about 5% of the total volume. With such a low intensity of measurement, the information obtained from the small end of a log or from the irregular butt end as required by Newton's or Smalian's method is of poor value compared to that of the midpoint.

CONCLUSIONS

This investigation has shown that the cubic spline curve can be useful to provide a "smooth" function of cross-sectional area against length for a tree with known values of cross-sectional area at various points along the stem. The estimate of total volume derived from the integral of the curve is not as good as that estimated from Newton's formula or Romberg's method, but better than that estimated from Smalian's formula.

This makes the curve useful for trees which have been measured at irregular intervals along the stem where Newton's formula or Romberg's method could not be used. A drawback of the curve is that the very property of "smoothness" may make it inappropriate when applied to those profiles of malformed stems with rates of taper which abruptly change.

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