Letter to the Editor

LOG PARAMETERS: LENGTH, DIAMETER, TAPER, FORM

Sir,

Dr Whyte (1971) recommends in his article on the sectional measurement of trees that diameter observations be restricted to the mid-internodes. Tree volume is obtained as the sum of component log volumes.

Dr Whyte's major conclusion relates to his Experiment 1. A tree is segmented into approximately cylindrical frusta. It matters little whether these sections are regarded more precisely as being of paraboloidal, conical, or neiloidal form when the total volume is calculated. He recommends the intermediate formula and a 10% maximum difference in end-diameters.

A typographical error in Dr Whyte's paper occurs in line 4, p.75, the intended formula undoubtedly being:

$$v = \tfrac{1}{4}\pi \cdot L \cdot (Dd + (D-d)^2/(r + 1)),$$

where L = length of frustum,
d = small-end diameter,
D = large-end diameter,
r = a constant,

or:

$$v = v_c \ (1 + t + \alpha t^2),$$

where $v_c = \frac{1}{4}\pi \ Ld^2$,
$t = (D-d)/d$,
$\alpha = {}^1/(r + 1)$.

With appropriate constant this formula gives the exact volume of any paraboloidal frustum ($\alpha = \frac{1}{2}$), any conical frustum ($\alpha = 1/3$), or ($\alpha = 0$) any frustum generated by rotating a segment of the rectangular hyperbola about an asymptote.

An increment, $\Delta\alpha$, in the parameter α produces the absolute effect Δv in the volume estimate, where:

$$\Delta v = v_c t^2 \ \Delta\alpha.$$

For small t the relative disparity between conical and paraboloidal volume estimates increases as the *square* of the fractional difference in end diameters. A *variable bias* of similar order is expected whenever a form hypothesis is arbitrarily accepted. In practice this would constitute a disincentive to reduce the per tree sampling intensity.

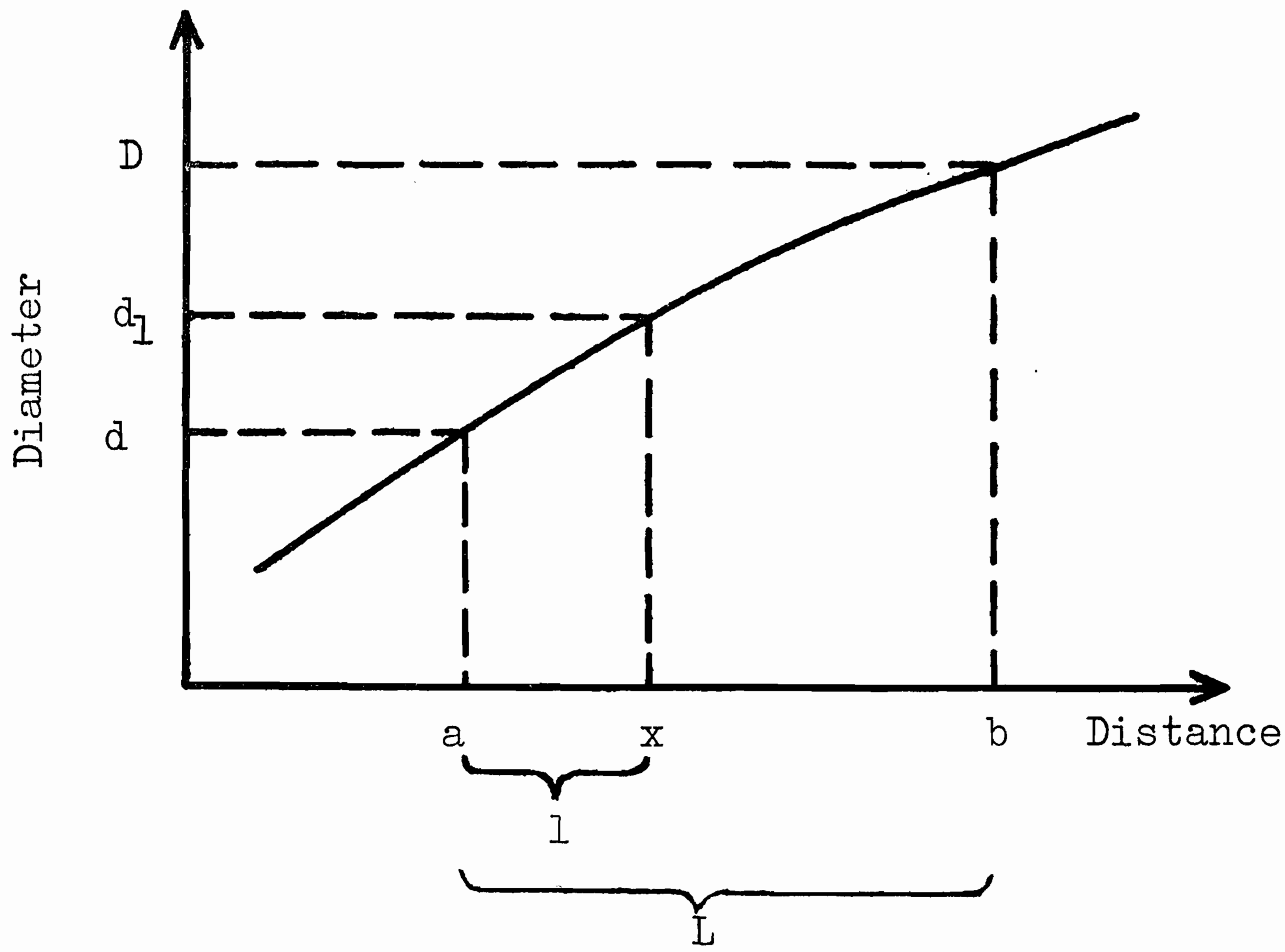
I believe that the generality of the above volume formula does not extend far beyond the three instances cited, if at all. In particular there does not exist a solid of revolution (the cylinder trivially excepted) whose volume between *arbitrary* cross-cutting planes is identically $1/16 \ \pi \ L \ (D + d)^2$.

A method for calculating the distance from the small end of a frustum to an arbitrary diameter, intermediate between given end-diameters, is outlined in Fig. 1. A frustum volume integral of the type:

$$v = v_c H \ (s),$$

where H is some function of s,
$s = D/d$,

must be postulated. Example (Smalian's formula): $H = \frac{1}{2}(1 + s^2)$. The following restriction has an important consequence. Initially the functional H is assumed only



Postulate: $\int_a^b d_1^2 \quad dx \sim (b-a)\, d^2\, H(s)$

Differentiate $\quad D^2 \sim d^2\, H + (b-a)\, d^2\, H' \,.\, ds/db$

w.r.t. b: $\quad s^2 = (D/d)^2 \sim H + (b-a)\, H' \,.\, ds/db$

Separate variables
and integrate:

$$\int_{a+l}^{a+L} \frac{db}{b-a} = \ln\left(\frac{L}{l}\right) \sim \int_{d_1/d}^{D/d} \frac{H'\,ds}{s^2 - H}$$

Exponentiate: $\quad l \sim L \exp\left\{ -\int_{d_1/d}^{D/d} \frac{H'\,ds}{s^2 - H} \right\}$

Calculate three distinct points on the graph of this function and inspect whether the sectional volumes indicated by the postulated formula are consistent. If a counter example does not come to hand try to obtain a proof for the inverse problem.

FIG. 1—Determination of the diameter-distance graph associated with a given volume formula of the type $v = v_c H(s)$ where $s = D/d$.

to apply to frusta of *constant* small-end diameter. H may be *compulsorily modified* when the resulting taper function is integrated to obtain the volume between *arbitrary* end diameters. (Application of the method to: $v = v_c (1 + t + \alpha t^2)$ yielded the essentially negative results of the preceding paragraph.)

Several authors (e.g., Prodan, 1965; Grosenbaugh, 1966; Whyte, 1971) have discussed the taper function:

$$y^2 = px^r,$$

where y = radius at distance x from an apex; p and r are positive constants.

The fact that the apex is remote from the small end of a log is initially a source of difficulty. Nevertheless a frustum volume integral may be obtained on the condition that the end diameters are positive and different:

Define $z = (D/d)^{2/r}$,

then $v = \frac{1}{4}\pi L (z D^2 - d^2)/(r + 1)(z - 1)$

$= v_c (z^{r+1} - 1)/(r + 1)(z - 1).$

When r is an integer the ratio v/v_c will be recognised as the arithmetic mean of the geometric series: $1 + z + z^2 + \ldots + z^r$.

Dr Prodan gives individual formulae for the frusta of paraboloids (r = 1), cones (r = 2), and neiloids (r = 3). Numerical data pertaining to the volume effects of form assumptions are given in Table 1.

TABLE 1—Comparative volumes of circular frusta

Fractional diff. in end-diameters: t = (D-d)/d		Volume of frustum* Par.	Con.	Neil.	—
	α:	½	1/3	—	0
	r:	1	2	3	—
0.0+		1.000	1.000	1.000	1.000
0.1		1.105	1.103	1.103	1.100
0.2		1.220	1.213	1.211	1.200
0.3		1.345	1.330	1.325	1.300
0.4		1.480	1.453	1.444	1.400
0.5		1.625	1.583	1.569	1.500
0.6		1.780	1.720	1.700	1.600
0.8		2.120	2.013	1.977	1.800
1.0		2.500	2.333	2.277	2.000
1.2		2.920	2.680	2.598	2.200
1.5		3.625	3.250	3.121	2.500
2.0		5.000	4.333	4.102	3.000

$$* v = \begin{cases} v_c (1 + t + \alpha t^2); & \alpha = \frac{1}{2}, 1/3, 0, \\ v_c (z^{r+1} - 1)/(r + 1)(z - 1); & r = 1,2,3, \end{cases}$$

where $v_c = \frac{1}{4}\pi Ld^2 = 1.000$,

$z = (D/d)^{2/r}$.

Dr Whyte's Experiment 2 relates to the interpolation of diameter at $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$

of the log length. Simpson's Rule (*see* McCalla, 1967, p277) may be invoked to obtain the mid-diameter of a frustrum with taper function:

$$y^2 = a_0 + a_1 x + a_2 x^2 + a_3 x^3,$$

where a_0, a_1, a_2, a_3 are constants, when the volume is a known function of length and end-diameters. The paraboloid, cone, and neiloid are perfect candidates. (A converse is that "Newton's formula" for calculating log cubic contents is robust to variations in log form.) The method does not yield the diameter d_l at an arbitrary intermediate position distant l from the small end.

I have derived the interpolation formula:

$$(d_l/d)^2 = (1 + (z-1)\ l/L)^r$$

appropriate to the taper function $y^2 = px^r$.

A difference of opinion with Dr Whyte on the choice of formula is demonstrated by comparative values for row 1 of his table 2:

Whyte:	100	125	112.5	106.2	103.1	117.7	112.5	108.8
McDonald:	100	125	106.80	106.25	106.07	113.19	112.50	112.27

A consequence of Dr Whyte's proposition (p.76, Experiment 2) that: "The absolute difference in interpolated diameter at a given point is identical, for a given difference between small- and large-end diameter, irrespective of small-end diameter" is that, in the case of the paraboloid, one vertex is displaced from the axis (Fig. 2).

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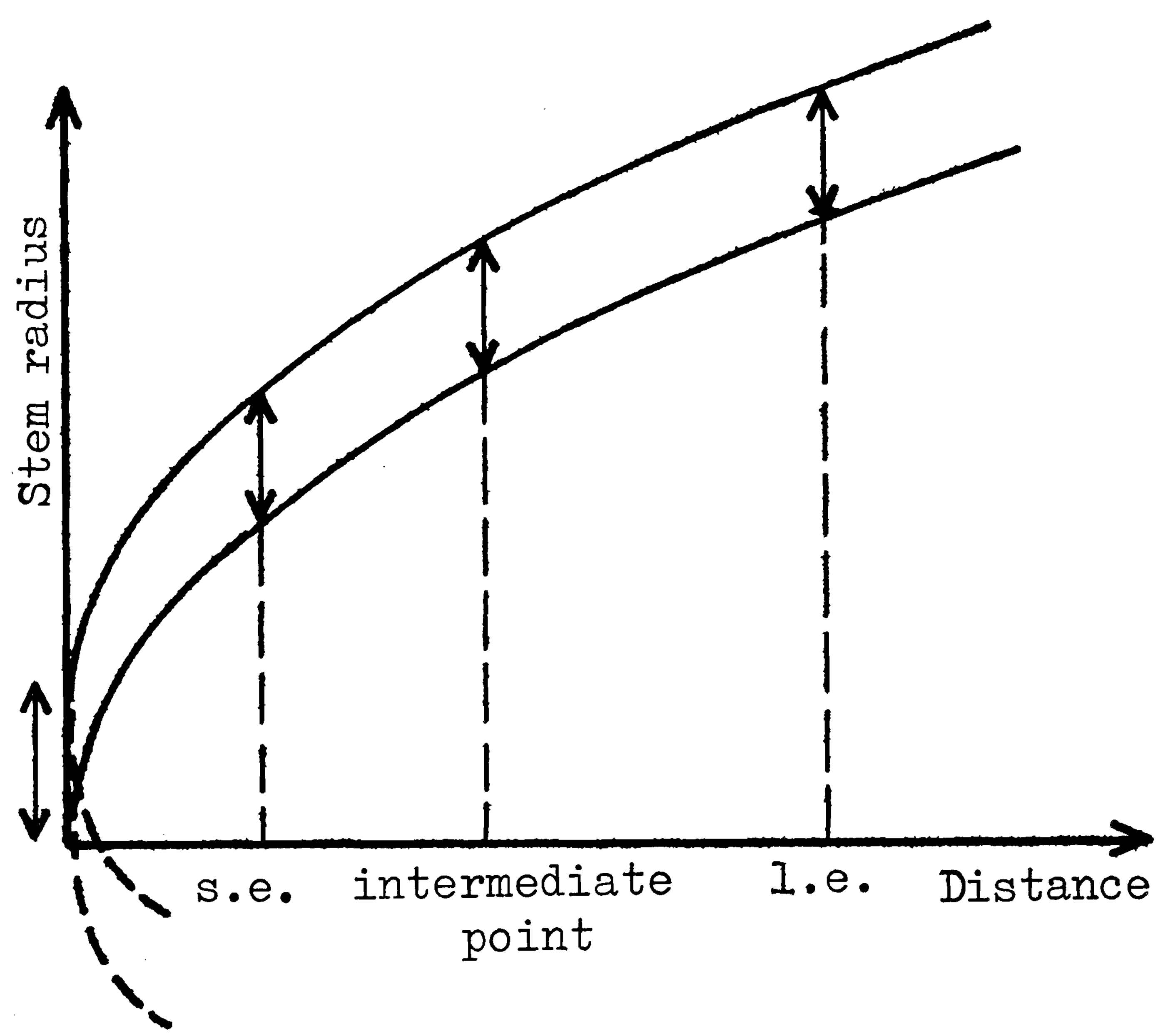

FIG. 2—**Reductio ad absurdum** on Whyte's result of his Experiment 2.

GROSENBAUGH, L. R. 1966: Tree form: definition, interpolation, extrapolation. **Forestry Chronicle 42 (4):** 444-57.

McCALLA, T. R. 1967: "Introduction to Numerical Methods and FORTRAN Programming". John Wiley & Sons, New York, London, Sydney, 359pp.

PRODAN, M. 1965: "Holzmesslehre". J. P. Sauerlander's Verlag, Frankfurt am Main, 644pp.

WHYTE, A. G. D. 1971: Sectional measurement of trees: a rationalised method. **New Zealand Journal of Forestry Science. 1 (1):** 74-9.

Letter to the Editor

MEASUREMENT OF TREES: A REJOINDER

Sir,

I appreciate your affording me the opportunity to clarify points raised by Mr McDonald (1971, this issue, pp 240-4) on my paper on sectional measurement of trees (*N.Z. Jl For. Sci. 1 (1):* 74-9). Your policy of encouraging discussion on papers is most welcome.

Mr McDonald is, of course, quite correct in identifying the misprint in line 4, page 75, but I understand you are noting this elsewhere [Corrigendum, p246]. He is correct, too, in observing the inappropriateness of my formula for interpolating diameters within a paraboloidal or neiloidal frustum with two positive, non-zero diameters. The formula used in my paper is correct, for cones, but for the other two curves only when the small-end diameter is zero, and not when frusta with small-end diameters greater than zero are being considered. I apologise for my error and am grateful to Mr McDonald for pointing it out.

Luckily, this error does not effect the important theses, that the greater the difference in end diameters, given any one small-end diameter, the greater is the difference between interpolated diameters among the three curves, and also the smaller becomes the percentage error as small-end diameter increases, for a given difference in end diameters. But, this need to assume a certain shape along any part of the tree stem in order to interpolate diameters may well be superseded by techniques such as those being developed in North America by, for example, Kozak *et al.* (1969) as explained in a previous paper (Whyte, 1971). Thus, it is much more important, with the methods of processing and analysing data now available, to have reliable, representative, and consistent measurements of diameter at known heights in a tree and to use least-squares regressions such as:

$$d = D\sqrt{b_0 + b_1(h/H) + b_2(h^2/H^2)}$$

to predict diameters at any chosen height (*see* Kozak *et al.*, 1969, p.280) in a given population of trees, where d is diameter inside (or outside) bark at any given height, h, above ground; D is diameter at breast height over bark; H is total height of tree and b_i are computed least-squares coefficients.

I am convinced that this approach is eminently more promising, as it dispenses with the need to make any assumption about the form between two heights on the