

## COMPATIBLE TREE VOLUME AND VARIABLE-FORM STEM TAPER MODELS FOR *PINUS RADIATA* IN TASMANIA

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### ABSTRACT

A tree volume model based on Schumacher's form and a compatible, segmented, and variable-form stem taper model were found to be the best models for predicting volume, volume and diameter to a given height, and volume to a top diameter limit for *Pinus radiata* D. Don sample trees. Schumacher's tree volume model was modified by incorporating DBH to total height ratio ( $R$ ) and mean annual DBH increment (DBH/age) as extra predictor variables. The variable-form stem taper model incorporated DBH,  $R$ , and DBH/age as extra predictor variables to the average-form model. The model predicts increasing butt swell with increasing DBH and increasing upper-stem taper with increasing DBH/age. Compatibility enforced algebraically gave more precise and less biased predictions of volume in the butt section compared to the empirically-compatible version of the taper model. The model suffered from bias when predicting diameter near the tip but bias was not a problem in volume prediction.

A random coefficients regression procedure was used to generalise the taper model from an average-form to a variable-form model.

**Keywords:** taper; tree volume; variable-form; random coefficients regression; *Pinus radiata*.

### INTRODUCTION

Models to predict tree volume, merchantable volume defined to either a given height or diameter limit, and diameter to a given height are required for inventory and growth and yield simulators. Compatible tree volume and stem taper models form a useful class of models to predict these quantities (Demaerschalk 1973; Goulding & Murray 1975; Van Deusen *et al.* 1982; Byrne & Reed 1986; McClure & Czaplewski 1986). The tree volume model in general predicts tree volume from a simple function of DBH and total height while the taper model predicts stem diameter given height, total height, and either DBH or estimated tree volume (i.e., from the tree volume model). Volumes to a given height or diameter limit are obtained by integrating the taper model. Compatibility means that on integration of the taper model from ground to tip the volume produced is the same as that produced from the tree volume model. Compatibility can be enforced algebraically in a number of ways: by recovering

parameter values of the tree volume model using the estimated parameters from the fit of the taper model (Byrne & Reed 1986), by estimating any "free" parameters of the taper model given estimated parameters of a particular form of tree volume model (Demaerschalk 1973; Van Deusen *et al.* 1982), or by constraining the parameters of, and incorporating estimated tree volume in, the taper model (Goulding & Murray 1975; McClure & Czaplewski 1986).

In another development, segmented taper models (which use separate equations to describe each of several segments of the tree bole and then join or spline these segments together at points defined by join-point parameters) were used by Max & Burkhart (1976) and Cao *et al.* (1980). Cao's model, which is a segmented version of Goulding & Murray's compatible fifth-degree polynomial taper model, was not constrained to be compatible but empirically they found it to be very close to compatible with tree volume estimated from the combined variable model. Byrne & Reed (1986) developed a version of Cao's segmented model which is algebraically compatible with the constant form-factor tree volume model. Compatibility was enforced by incorporating a form factor parameter in Cao's model so that the tree volume model's single parameter estimate was obtained in the fit of the taper model. McClure & Czaplewski (1986) introduced a constraint on the parameters of Cao's model to make it algebraically compatible with any form of tree volume model.

In a recent development Newberry & Burkhart (1986) used a variable-form taper model based on Ormerod's (1973) two-parameter taper model for the main stem of *Pinus taeda* L. where main stem was defined as the bole segment below the crown and above the point affected by butt swell. Although we are concerned here only with whole-tree models, their study is relevant because the model developed was variable-form. They followed Gray's definition of form as the characteristic shape of the solid describing the bole, which generalises to more than one solid for segmented models, while taper is defined as the rate of narrowing in diameter in relationship to the increase in height for a solid of given form. By variable form it is meant that a base model is defined which, when fitted to a sample of trees, predicts the taper for the tree of average form. The variable-form model allows the parameters of the base model to vary as functions of tree and/or stand variables. A random coefficients, also called varying-parameter (Biging 1985), regression procedure was used to construct the above model.

The objective of this study was to develop tree volume and stem taper models to predict under-bark (ub) volumes and diameters for any site in Tasmania and/or any particular management practice such as planting spacing, thinning, and pruning. The direct effects of management practices and their indirect effect, via stand and tree variables, on tree volume and stem taper were to be tested and quantified in the model where the necessary data were available and where these effects were statistically significant. Regional differences were to be incorporated indirectly through their effect on stand and tree variables.

To achieve this objective the usefulness of features of the above taper models was tested. Attention was restricted to Cao's segmented model as the base model since it has been found to be a good multi-purpose taper model (Cao *et al.* 1980; Byrne &

Reed 1986). Gordon (1983) and Lowell (1986) have both attempted to improve Goulding & Murray's (1985) polynomial taper model; however, Cao's model was preferred here since its segmented nature facilitates the development of a variable-form model because the parameters have simple interpretations. This allows prior notions of how form varies to be incorporated in the modelling process. Also, Goulding & Murray's polynomial taper function requires a solution to a fifth-degree polynomial in order to obtain volume to a merchantable diameter limit, whereas this volume is much simpler to calculate using Cao's model since only the solution of a quadratic is required.

The usefulness of including in the tree volume model tree variables other than DBH and total height as well as stand variables was also examined.

A two-stage, random coefficients, modelling procedure similar to that used by Newberry & Burkhart (1986) was used to construct the variable-form taper model.

### Notation

$b_i$	= tree volume regression coefficients
$\alpha_i, \beta_i, \beta_{ij}$	= stem taper regression coefficients
$D$ or DBH	= diameter over-bark (ob) at breast height (cm)
$D_u$	= diameter (ub) at breast height (cm)
$H$	= total height (m)
$b$	= height above ground to top diameter $d$
$d$	= top diameter (ub) to height $b$ (cm)
$K$	= $\pi/[4(100^2)]$ constant to convert DBH <sup>2</sup> to basal area (m <sup>2</sup> )
$V$	= volume to tip (ub) (m <sup>3</sup> )
$v$	= volume to height $b$ (ub) (m <sup>3</sup> )
$z$	= $(H-b)/H$ relative height from tip to top diameter
$R$	= ratio $D/H$ (cm/m)
$T$	= age (years)
$MD$	= mean annual diameter increment $D/T$ (cm/year)
$y$	= $d^2KH/V - 2z$ the dependent variable for stem taper regressions where $V$ is the estimate from the tree volume model.

### DATA

The data used in this study consisted of 2224 sample trees of which 939 were trees with under-bark diameters measured directly at each measurement point on the stem by removing the bark. For nine out of 10 of the remaining sample trees, over-bark diameters only were measured at every point and under-bark diameters measured at 0.6 m, breast height, and (in most) at a fixed point at height 6.1 m. For the one in 10 tree, which was randomly selected, the bark thickness at each measurement point on the stem was obtained using a bark gauge. To obtain under-bark diameters where they were not measured directly predictive equations were used. These equations were developed using the random subsample data and a multiple linear regression of double bark thickness ( $DBT$ ) at measurement height on  $DBT$  at breast height and 6.1 m as well as diameter over bark at breast height and measurement height, total tree height,

and age. Tests of accuracy showed that the volumes obtained by this method demonstrated negligible bias and were within 2% of those obtained by actually measuring the under-bark diameters (P. R. Lawrence, "Metric volume tables for *Pinus radiata*", Forestry Commission report). The Smalian formula was used to calculate the under-bark volume between the points of measurement. Volume between ground and the first measurement point at 0.6 m was obtained by linearly extrapolating the under-bark diameter to the ground using diameter at 0.6 m and 1.3 m.

Information available for the sample trees is summarised in Table 1. Information on thinning is restricted to the number of times stands from which the sample trees were drawn had been thinned, which ranged from 0 to 6, with 54% of the trees from unthinned stands and a further 42% from stands thinned at most twice. The thinnings were usually frequent and at later ages with light removal of growing stock corresponding to pre-1973 (i.e., "old") regimes in Tasmania. All the sample trees were measured before 1973 after which time new intensive management regimes were introduced (Neilsen & Davis 1985). Square planting spacings of 1.83 m and 2.44 m accounted for 19% and 65% respectively of the trees for which spacing was known (i.e., there were 91 missing values).

TABLE 1—Summary of sample tree data

Tree/stand variable	Mean	Minimum	Maximum	No. of missing values
DBH ( <i>D</i> )	27.8	9.1	78.0	0
Height ( <i>H</i> )	21.8	8.5	50.0	0
Double bark thickness (cm)	3.2	0.4	12.2	0
Tree volume ( <i>V</i> )	0.66	0.03	6.78	0
Age ( <i>T</i> )	17.0	8.0	53.0	0
Site Index* (m)	30.3	16.4	39.4	358
Mean dominant height (m)	24.0	10.4	46.6	237
Pruned height† (m)	4.9	1.2	9.9	744
No. of measurement points	11.2	5	19	0

\* Mean dominant height at age 20.

† Pruned trees; 485 trees were unpruned.

## BASE MODELS

The models are described below in their base form by which it is meant that only basic tree variables of  $V$ ,  $D$ ,  $D_u$ ,  $H$ , and within-tree variables  $d$ ,  $b$ , and  $z$  are used as predictor variables (i.e.,  $V$  in the stem taper model). In the stem taper model the base model could also be called the first-stage (*fs*) model since it is the model fitted individually to each sample tree in the first stage of the modelling procedure. Generalisation of the models to include other tree, tree-level (i.e., combinations of tree and stand variables), and stand variables is left to the Results section.

### Tree Volume Model

Two base models were considered here. Firstly, the simple combined variable model recommended by Burkhart (1977) gives:

$$V = b_1 + b_2 D_u^2 H \quad (1)$$

The constant form factor model is equivalent to Eqn (1) with  $b_1$  set to zero (i.e., fitted through the origin).

Secondly, Schumacher's model was tried where

$$V = b_0 D_u^h H^h \quad (2)$$

### Stem Taper Model

The model used here was given by Cao *et al.* (1980) and is a segmented version of Goulding & Murray's (1975) compatible taper equation based on a fifth-degree polynomial in  $z$ . By using three segments, Cao's model requires five parameters to be estimated so that the model is given by:

$$d^2(z)KH/V - 2z = \beta_1(3z^2 - 2z) + \beta_2(z - \alpha_1)^2 I_1 + \beta_3(z - \alpha_2)^2 I_2 \quad (3)$$

$$\text{where } \begin{array}{l} I_i = 1 \quad \text{for } z \geq \alpha_i \quad i = 1, 2 \\ \quad = 0 \quad \quad z < \alpha_i. \end{array}$$

The linear parameters to be estimated are the  $\beta_i$  while  $\alpha_1$  and  $\alpha_2$  are nonlinear, join-point parameters which, combined with the indicator variables  $I_i$ , separate the segments of the model on the relative height axis. Proceeding from the tip ( $z = 0$ ) the first component on the right-hand side of Eqn (3) contributes to all segments, for  $\alpha_2 > \alpha_1$  the second component to the middle and butt segments, and the third component contributes only to the butt segment. When  $\alpha_2 < \alpha_1$  the roles of the second and third components are reversed. This model allows three basic shapes in the  $d^2(z)$  and  $z$  relationship considering  $KH/V$  as a scaling factor. In application, only an estimate of  $V$ ,  $V$ , is available for use on the left-hand side of Eqn (3) whereas for the fit of Eqn (3) either  $V$  or  $V$  can be used,  $V$  obtained from the tree volume equation. However, for the following  $V$  will be used so that the precision of predictions of  $v$ ,  $b$ , or  $d$  is that expected in application.

McClure & Czaplewski (1986) showed how compatibility can be enforced by constraining the parameters of Eqn (3) so that

$$\beta_2/\beta_3 = -[(\alpha_2 - 1)/(\alpha_1 - 1)]^3 \quad (4)$$

The constraint given by Eqn (4) was incorporated in Eqn (3) here, as was done by McClure & Czaplewski, by expressing  $\beta_3$  in terms of  $\beta_2$ ,  $\alpha_1$ , and  $\alpha_2$  and substituting the result for  $\beta_3$  in Eqn (3).

Volume to height  $b$  can be recovered by integration of Eqn (3) with respect to  $z$ . Volume to under-bark diameter  $d$  can be obtained by first calculating the height to diameter  $d$  using Eqn (3) and then integrating Eqn (3) to this height. The formulae for calculating height and volume to a given diameter are given by McClure & Czaplewski (1986).

The constraint that  $d = 0$  when  $b = H$  is implied in Eqn (3). Unfortunately it does not appear possible to constrain  $d$  to equal  $D_u$  when  $b = 1.3$  m with an equation of the form of (3). Cao *et al.* (1980) did not impose the constraint given by Eqn (4) but allowed all five parameters to vary freely. They relied on the relationship between

the estimates of parameters to make the model "empirically" compatible with predicted tree volume. In four examples that they studied  $v_H/V$ , where  $v_H$  is  $v$  predicted at  $b = H$ , was 0.9896, 0.9943, 0.9942, and 0.9959. This empirical compatibility is the result of the least squares fit which attempts to predict stem taper as closely as possible so that on integration  $v_H$  closely approximates  $V$ . However, by using  $V$  in Eqn (3) and the constraint given by Eqn (4) then compatibility is enforced algebraically with  $v_H/V = 1$  for any form of tree volume model. Some loss of predictive ability for  $y$ , where  $y = d^2KH/V - 2z$ , and thus  $d$  is always incurred by constraining the parameters as in Eqn (4) and it must be decided if this loss is offset by the advantages of compatibility. This is discussed in the Results section.

Note that at this stage Model (3) is an average-form model since given  $V$ ,  $H$ , and  $D_u$  (i.e., since  $V$  is a function of  $D_u$ ) then  $d$  depends only on  $z$ . Investigation of a variable-form model can proceed by allowing the parameters in Eqn (3) to vary by tree. Newberry & Burkhart (1986) used such an approach using as base model Ormerod's taper model involving a form and a taper parameter.

### STATISTICAL METHODOLOGY

The estimation of, and significance tests for parameters in the tree volume models are relatively straightforward. Model (1) is linear (in the parameters) and the parameters of Model (2) can be rearranged as follows:

$$V = \exp(b_0 + b_1 \ln D_u + b_2 \ln H) \quad (5)$$

where  $b_0 = \ln b_0$

If logarithms are taken on both sides of Eqn (5) the resulting model is linear. The logarithmic transformation also helps to stabilise the variance of  $V$  since the conditional variance of  $V$ ,  $Var(V|D,H)$ , is well known to increase with  $V$ ,  $E(V|D,H)$  (i.e., the expected value of  $V$ ) and  $D^2H$ . The method used here to account for non-constant conditional variance was to assume the following variance function (i.e., variance functionally related to the expected value)

$$Var(V|x) = \Phi \mu^2 \quad (6)$$

where

$\mu = E(V|x)$ ,  $x$  is a vector of prediction variables such as  $D$ ,  $H$ ,  $R$ , and  $\Phi$  is a dispersion parameter.

The parameters in Models (1) and (5) can be fitted using iteratively reweighted least squares using the variance function (6) and the algorithm of Nelder & Wedderburn (1972) which is equivalent to assuming a gamma distribution for  $V$ . The theory behind this methodology is based on quasi-likelihood which uses a statistic called the deviance, which is the maximum likelihood (ML) generalisation of the residual sum of squares, to test the significance of adding or subtracting terms from the model (McCullagh & Nelder 1983). The advantage of this method is that no bias is incurred in predicting  $V$  whereas bias is incurred when re-transforming to the original scale of volume after using  $\ln V$  as the dependent variable (Flewellling & Pienaar 1981).

For the construction of the stem taper model the dependent variable used in regressions was  $y$  where in the calculation of  $y$ ,  $V$  is the predicted volume from the final form of the tree volume model. Graphical inspection of residuals indicated no heterogeneity of variance so no weighting of regressions was required.

Construction of a variable-form taper model used a two-stage, random coefficients regression procedure whereby in the first stage the base model (3) was fitted separately to each tree and the resulting parameter estimates considered, in the second stage, as a set of five, correlated random variables with expected value given by corresponding second-stage ( $ss$ ) submodels. Each of these  $ss$  submodels may incorporate tree, tree-level, and stand variables (called  $ss$  covariates for the following). Such two-stage modelling procedures are common in forestry applications with previous work by Ferguson & Leech (1978), Davis & West (1981), West *et al.* (1984), and Newberry & Burkhart (1986) being most relevant to the discussion here. These authors used a generalised least squares ( $gls$ ) procedure to estimate the  $ss$  parameters. They also compared  $gls$  to ordinary least squares ( $ols$ ) estimation. Davis & West (1981) and Newberry & Burkhart (1986) concluded that the  $gls$  solution is not entirely satisfactory or worthwhile given the effort required to obtain it.

An alternative two-stage regression approach to that used by the above authors and described by Candy (1988) was used here. This approach uses a sequential sub-model identification procedure combined with  $ols$  estimation.

## RESULTS AND DISCUSSION

### Tree Volume Model

The fit of Models (1) and (5) is given in Table 2. Examination of residuals revealed a trend with  $D$  for Model (1). The residuals were negatively biased for  $D$  approximately 40 cm and greater. This trend was more severe when the constant form model was fitted. There was no such trend in the residuals from Eqn (5). Pearson residuals (McCullagh & Nelder 1983) indicated that the variance function (6) was appropriate. The results for Eqn (5) indicated that Schumacher's equation was the best form of the model to develop further. Warner (1988) also found Schumacher's model to be the best of those tested for *P. radiata* in northern Tasmania.

TABLE 2—Fit statistics for tree volume models

Model	No. of parameters	Residual deviance*	Nested model†	Stat.‡	S.E. (%) ( $\hat{V}$ )§
Combined variable (1) <i>CV</i>	2	18.21	<i>CF</i>	857¶	9.1
Constant form factor (1) <i>CF</i>	1	25.24			10.7
Schumachers (2) <i>S</i>	3	16.32			8.6
<i>S</i> plus <i>R</i> , <i>MD</i> (7)	5	15.40	<i>S</i>	133¶	8.3

\* Maximum likelihood equivalent of the residual sum of squares which gives an estimate of  $\Phi$  as the residual deviance divided by residual degrees of freedom (i.e., 2224 - number parameters).

† Model involving a subset of parameters.

‡ Statistic to determine if the change in residual deviance is due to chance calculated as the absolute value of this change divided by  $\hat{\Phi}$  and compared with a chi-square with degrees of freedom equal to the absolute value of the difference in number of parameters.

§ Percentage variation of  $\hat{V}$  about  $V$  given by  $100 \sqrt{\hat{\Phi}}$ .

¶ Probability <0.0001.

After examination of residual plots, using  $\ln V - \ln V$  as residuals, against each covariate the final model involved  $R$  and  $MD$  as follows, with parameter estimates and their standard errors given in Table 3 and fit statistics given in Table 2.

$$V = \exp[b_0 + b_1 \ln(D_u) + b_2 \ln(H) + b_3 \ln(R) + b_4 \ln(MD)] \quad (7)$$

The percentage standard error (i.e., percentage of predicted values) of predicted tree volume (about the fitted regression) given by  $100\sqrt{\hat{\Phi}}$ , where  $\hat{\Phi}$  is the residual mean deviance (i.e.,  $ML$  equivalent of the residual mean square), was 8.3%. The percentage standard error was less for the under-bark (6.6%) than for the over-bark sample trees (9.4%). The mean percentage bias calculated as the mean of  $100(V - \hat{V})/\hat{V}$  was -0.002% with a value of 2.2% for the under-bark and -1.6% for the over-bark sample trees.

The above models were fitted using *GENSTAT* (Numerical Algorithms Group 1983).

TABLE 3—Parameter estimates for Models (7) and (9) (standard errors in parentheses)

Model	Parameter						
	$\beta_{10}$ $b'_0$	$\beta_{11}$ $b_1$	$\beta_{12}$ $b_2$	$\beta_{21}$ $b_3$	$\beta_{31}$ $b_4$	$\alpha_1$	$\alpha_2$
Tree volume	-9.8766 (0.0152)	1.7612 (0.0632)	1.1250 (0.0609)	0.1272 (0.0610)	-0.1067 (0.0099)		
Stem taper	-0.1821 (0.0311)	0.2731 (0.0377)	0.1844 (0.0208)	-0.0653 (0.0350)	0.8136 -	0.6511 (0.0408)	0.8495 (0.0137)

### Stem Taper Model

The first-stage analysis was carried out by fitting Model (3) to each sample tree allowing all five parameters to vary freely. The pooled residuals in diameter ( $d-d$ ) were then graphed against  $z$  and a trend of negative bias increasing from approximately  $z = 0.2$  to 0.0 (i.e., the tip) was observed. Goulding & Murray (1975) and Lowell (1986) observed a similar trend for their average-form model. In Model (3) it is the parabolic shape of Eqn (3), in terms of  $b$ , near the tip which over-estimates the actual more conical tree taper. Since it is not possible to retain this model's basic form while at the same time incorporating a conical tip segment it was considered that it was reasonable to persevere with Model (3) and to examine the effect of the above bias once the final model was fitted.

### Second-stage Analysis

Graphical examination of the  $f_s$  parameter estimates was carried out at this stage. In the join-point parameters it was found that some estimates of either  $\alpha_1$  or  $\alpha_2$  were outside the [0,1] range and for some trees both were outside this range. In the first instance Model (3) is reduced to two segments and in the second to a single segment. Since no trends between the  $\alpha_1$  and  $ss$  covariates which could explain this were observed and a submodel to predict the number of segments would add an extra, possibly unnecessary level of complexity to the model, a three-segment model was assumed.

When  $\alpha_1$  were both in the range [0,1] the  $f_s$  parameter estimates obtained above were graphed against  $ss$  covariates but no trends were obvious. The  $\alpha_1$  were then assumed to be independent of the measured covariates and the resultant submodels, each involving a simple mean of the  $\alpha_1$  for  $0 < \alpha_1 < 1$ , were substituted back into Model (3). This model was then refitted while allowing only the three linear parameters to vary by tree. The resulting conditional estimates of each  $\beta$  were graphed against  $ss$  covariates and the only trend observed was between  $\beta_1$  and each of  $R$  and  $MD$ . Incorporating these effects gives the first term of the model now as  $\beta_1(3z^2-2z)$  where

$$\beta_1 = \beta_{10} + \beta_{11}R + \beta_{12}MD.$$

The  $f_s$  parameter estimates were obtained using a special purpose FORTRAN program which used the Gauss-Newton method of nonlinear estimation.

From this point on the model was developed and estimated using a random subset of the data where a single  $d$ , and corresponding  $h$  and  $v$  values were randomly selected from each tree. Although the above analyses did not suggest any relationship between the  $\beta_2$  or  $\beta_3$  and tree size such a relationship would be expected. The relative size of  $\beta_2$  and  $\beta_3$  determines the predicted degree of butt swell since for  $\alpha_2 > \alpha_1$  increasing  $\beta_3$  relative to  $\beta_2$  increases butt swell while  $\alpha_2$  determines the point below which butt swell occurs. So to investigate the effect of DBH on butt swell the data were classified into four DBH classes ( $< 20$ ,  $20-30$ ,  $30-40$ ,  $> 40$  cm). The parameter estimates for each DBH class were then graphed against the class midpoints to help determine how DBH could be incorporated in Model (3). Unlike the previous  $ss$  graphical analysis of  $\beta$ , there was a trend with DBH for each of  $\beta_2$  and  $\beta_3$ . The following model, incorporating the best  $ss$  submodels for  $\beta_2$  and  $\beta_3$  of those investigated, was fitted to the random subset data

$$y(z) = \beta_1(3z^2-2z) + \beta_{21}D(z-\alpha_1)^2I_1 + \beta_{31}D(z-\alpha_2)^2I_2 \quad (8)$$

where  $I_i$  are as given for Model (3).

Note that the formulae, based on Eqn (8), for calculating diameter and volume to a given height as well as those for calculating height and volume to a given diameter are the same as those for Model (3) but with  $\beta_1$  replacing  $\beta_1$ ,  $\beta_{21}D$  replacing  $\beta_2$ , and  $\beta_{31}D$  replacing  $\beta_3$ .

A compatible version of Model (8) was fitted using the constraint given by (4) imposed on  $\beta_{31}$ . The form of the compatible model is thus

$$y(z) = \text{as for Model (8)} \quad (9)$$

$$\text{where } \beta_{31} = -\beta_{21}/[(\alpha_2-1)/(\alpha_1-1)]^3.$$

The parameter estimates and their standard errors for Model (9) are given in Table 3. The average-form model, Model (3), in both non-compatible and compatible forms, was also fitted for comparison and the statistics of fit are given in Table 4 along with those for Eqn (8) and (9). Eqn (9) has a 20.5% lower residual sum of squares (RSS) in terms of diameter than the compatible average-form model but has a slightly higher (3.2%) RSS than Model (8). A similar slight loss of predictive ability was incurred by making the average-form model compatible.

TABLE 4—Fit statistics for stem taper models

Model	No. of parameters	RSS*		Percentage variance of $y$	S.E. ( $\hat{d}$ )†
		$y$	$d$		
Average form (3)	5	48.33	2754.0	74.8	1.11
Average form (3)‡	4	48.55	2877.0	74.7	1.14
Variable form (8)	7	38.21	2311.5	80.1	1.02
Variable form (9)	6	39.86	2387.9	79.1	1.04

\* Residual sum of squares

†  $\sqrt{(\text{variance})}$  of  $\hat{d}$  about  $d$  (cm)

‡ Compatible version

Nonlinear *ols* estimation for the above models was carried out using *GENSTAT*.

The "empirical" compatibility of Eqn (8) was examined by integration between the ground and the tip as described earlier and comparing this volume to  $V$ . The ratio of  $v_H/V$  was tabulated as a percentage across DBH classes. Over all classes the average was 97.99%; however, this figure ranged from 98.82% for DBH < 20 cm down to 95.23% for DBH > 60 cm. The empirical compatibility of Model (3) is constant by nature of the model and was found to be 96.91%. Clearly the least squares fit of the non-compatibility model sacrifices compatibility where data are sparse (above 40 cm) for goodness of fit in the range of DBH where most of the data lie.

Percentage bias and precision *versus* relative height for each of  $d$  and  $v$  given  $h$  and  $v$  given  $d$  for both Eqn (8) and (9) are given in Fig. 1. Only the results for the random subset data for the under-bark sample trees are shown in Fig. 1. The bias and precision, expressed as a percentage of prediction, for  $d$  were calculated as the mean and standard deviation, respectively, of  $100(d-\hat{d})/\hat{d}$  for each relative height class where 0.1 classes of  $z$  from 0 (tip) to 1 (ground) were used. The corresponding values for  $v$  given  $h$  and  $v$  given  $d$  were calculated in a similar fashion. The bias of  $d$  in the 0.0 to 0.1 relative height class is understated in Fig. 1. There were only nine sampled heights in this class if selections of total height (i.e.,  $z=0$ ), which have been included in Fig. 1 and for which bias of  $d$  is zero, are ignored and the bias was -19.8% for Eqn (8) and -26.1% for Eqn (9). However, bias and lack of precision of  $d$  near the tip have not resulted in large biases or poor precision in  $v$ , given either  $h$  or  $d$ , near the tip (Fig. 1). In fact the precision of  $v$  near the tip approaches that of  $V$  (6%) as would be expected. In terms of bias and precision for volume prediction, Eqn (9) is superior to Eqn (8) in the butt region with the exception of precision of  $v$  given  $d$  where both models give similar results. The poor precision of both models here is of little practical significance for merchantable volume prediction since volumes in the butt section are usually determined from a stump height. Over-all the mean precision of  $v$  given  $h$  for Model (9) for the under-bark sample trees, with corresponding results for over-bark sample trees given in brackets, and adjusting for a mean bias of 1.7% (-0.2%) was 5.7% (7.4%) while bias and precision for the compatible average-form model, Eqn (3) and (4), were 0.9% (-0.5%) and 7.1% (8.5%) respectively. The corresponding figures for  $v$  given  $d$  were 3.9% (-0.8%), 20.1% (25.1%) and

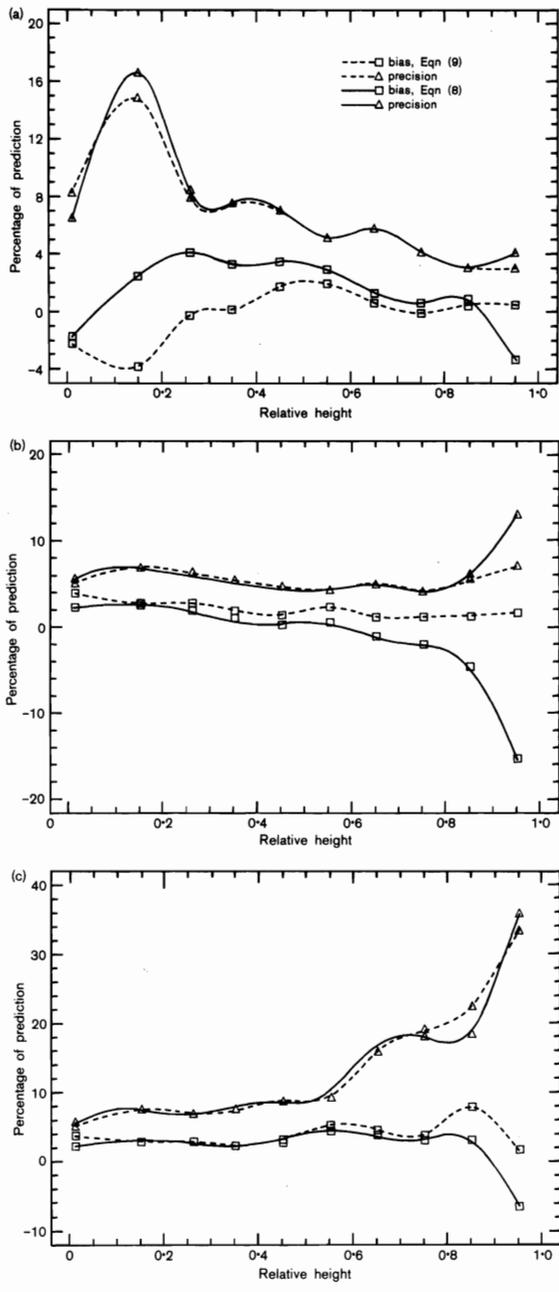


FIG. 1—Bias and precision of predictions of diameter and volume for under-bark sample trees.  
 (a) Diameter at a given height  
 (b) Volume to a given height  
 (c) Volume to a given diameter

3.1% (-0.2%), 24.1% (26.4%). So an improvement in precision in volume prediction as well as diameter has been achieved by developing the variable-form model.

Predicted taper for Models (8) and (9) for a range of DBH is compared in Fig. 2, from which it can be seen that the degree of divergence between these models increases with increasing DBH. This divergence causes the differences in predicted volume to increase with DBH and towards the base which may seem counter-intuitive given that the stem profiles in Fig. 2 converge again at the ground. The reason for this effect is that volume is obtained by integrating  $y$ , in terms of  $z$ , from the tip downwards so that differences in stem profile reflecting differences on the  $y$  versus  $z$  scales become cumulative downwards on the  $v$  versus  $h$  scales. This effect was also noticed when bias was calculated for each cell of the two-way DBH by percentage relative height classification. The bias of  $v$  given  $h$  for Model (8) for percentage  $z > 80$  varied from -4% for DBH < 20 cm to -26% for DBH > 60 cm while for Model (9) the range was 3% to -5%. The effect of varying DBH increment on form for fixed DBH is shown in Fig. 3. The interpretation of Fig. 3 is confused by the fact that the base model does not truly reflect actual taper near the tip but, taking this into account, taper appears to increase with increasing DBH increment in the top two segments. Investigation of the way changes in  $R$  change average form is not possible since for fixed DBH varying total height also changes average form because, as mentioned previously, given DBH and  $H$  Model (3) describes average form.

## CONCLUSIONS

The algebraically-compatible, variable-form, segmented stem taper model combined with the modified Schumacher tree volume model is the recommended system of models for the prediction of tree volume, stem taper, upper stem volumes to either a given height or diameter limit, and lower stem volume to a given height.

The ability to make any form of tree volume model compatible with Cao's model has the advantage that the best tree volume model can be selected. The combined variable and constant form factor models inadequately modelled tree volume compared to Schumacher's model. A slight but significant improvement in Schumacher's model was obtained by incorporating mean annual DBH increment and DBH/total height ratio.

Incorporating these two tree variables as well as DBH improved the fit of Cao's segmented taper model. The model's control of butt swell using DBH justifies the assumption of a three-segment model even for small trees with little or no butt swell for which a two-segment (i.e., no butt swell) model would be adequate. The use of DBH in this way simplifies the model by allowing one form of base model for the whole DBH range and is a logical extension of the average-form model. Further work is required to validate the predicted effect of DBH increment on taper.

Enforcing compatibility on the taper model is convenient since there is no need to try to reconcile two different estimates of tree volume (i.e., obtained either from the tree volume equation or by integrating the taper equation from ground to tip) as with the non-compatible model. Also, compatibility resulted in improved precision

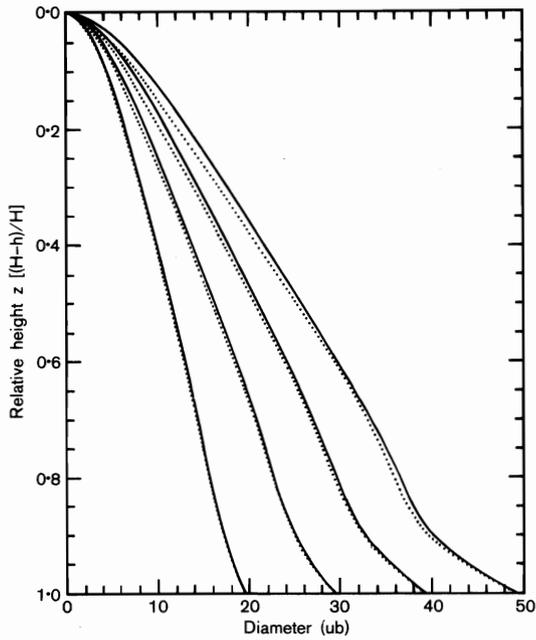


FIG. 2—Predicted stem taper for Models (8) ( . . . . ) and (9) ( ——— ) for a DBH of 20, 30, 40, and 50 cm.

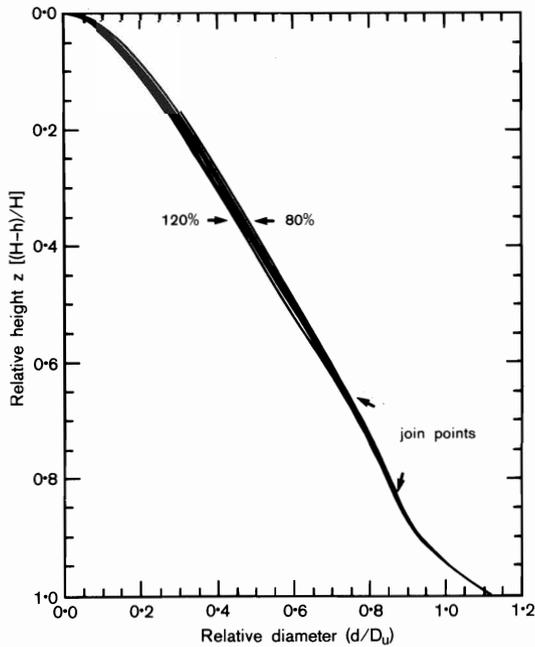


FIG. 3—Predicted stem profiles for Model (9) for a DBH of 40 cm and percentages of 80, 90, 100, 110, and 120 of the mean DBH increment of 2.1 cm/year.

and reduced bias of volume predictions in the butt section. The compatible model suffered a larger bias of diameter predictions near the tip but less bias in the butt than the non-compatible model. If estimates from the tree volume equation are unbiased then estimates of tree volume obtained by integrating the non-compatible taper equation will be biased. For large trees this bias was large, around 4.8%.

The direct effects of stand variables such as site index and management practices such as thinning have not been included in the model since their direct effect could not be detected; however, their indirect effect, via DBH/total height ratio and mean annual DBH increment, has been included in both tree volume and stem taper models.

The precision of predictions was in general poorer for the over-bark sample trees although bias was in general smaller and in the opposite direction to the under-bark sample trees. The difference in the bias of predictions between the two sets of sample trees was statistically significant ( $p < 0.01$ ) but it is not clear at this stage what effect these differences will have in application. The difference in bias between samples reflects variation in total tree volume and/or stem taper unaccounted for by the model. The selection of a sample tree to be measured either under-bark or over-bark was not determined randomly so the above differences could be due to regional variation. Empirical Bayes estimation, for example, as applied to nonlinear growth models by Berkey (1982) and Gertner (1984), could be used to regionalise the models but the added accuracy of such an approach would be at a cost in terms of simplicity of model application.

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